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WEAK CONVERGENCE OF SEMIGROUPS OF ASYMPTOTICALLY NONEXPANSIVE TYPE ON A BANACH SPACE

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1. Introduction

Let G be a semitopological semigroup, i.e., G is a semigroup with a Hausdorff topology such that for each $a\in G$ the mappings $s\to a\cdot s$ and $s\to s\cdot a$ from G to G are continuous. G is called right reversible if any two closed left ideals of G has non-void intersection. In this case, $(G, \geqslant)$ is a directed system when the binary relation "$\geqslant$" on G is defined by

$$t\geqslant s \text{ if and only if } \{s\}\cup\overline{Gs}\supseteq\{t\}\cup\overline{Gt},\quad s, t\in G.$$

Right reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups which are right amenable as normal semigroup (see [5, p. 335]). Left reversibility of G is defined similarly. G is called reversible if it is both left and right reversible.

In 1976, Kirk [8] introduced any non-Lipschitzian self-mapping which extends, in a sence, an asymptotically nonexpansive mapping inherited by Goebel and Kirk [3]; a continuous mapping $T: K\to K$, K a nonempty closed subset of a real Banach space X, is said to be of asymptotically nonexpansive type if for each $x\in K$,

$$\limsup_{n\to\infty}\{\sup[\|T^n x-T^n y\|-\|x-y\|] : y\in K\}\leqq 0.$$

Now, we introduce a semigroup of non-Lipschitzian self-mappings; let C be a nonempty closed convex subset of a real Banach space X with norm $\|\cdot\|$. A family $\mathcal{T}=\{T_s : s\in G\}$ of continuous mappings of C into C is said to be a right reversible semigroup of asymptotically nonexpansive type on C if the following conditions are satisfied:

(a) the index set G is a right reversible semitopological semigroup with the binary relation "$\geqslant$" defined as above;

(b) $T_{st}x = T_s T_t x$ for all $s, t \in G$ and $x \in C$;

(c) for each $x \in C$,
$\inf_s \sup_{t \geq s} \{\sup[\|T_t x - T_t y\| - \|x - y\|] : y \in C\} \leqq 0$;

(d) T is continuous with respect to the strong operator topology: $T_s x \to T_t x$ for each $x \in C$ as $s \to t$ in G.

Left reversible semigroup of asymptotically nonexpansive type is defined similarly. $\mathcal{T} = \{T_s : s \in G\}$ is reversible if it is both left and right reversible. For semigroups of another non-Lipschitzian self-mappings, see [1], [6], [7] etc.

For each $x \in C$, $\mathcal{T}(x) = \{T_s x : s \in G\}$ is called the orbit of x under $\mathcal{T}$ and a point $z \in C$ such that $\mathcal{T}(z) = \{z\}$ is called a common fixed point of $\mathcal{T}$. We denote by $F(\mathcal{T})$ the set of common fixed points of $\mathcal{T}$ and by $\omega_w(x)$ the set of weak subnet limits of the net $\{T_s x : s \in G\}$. We set $E(x) = \{y \in C : \lim_{s \in G} \|T_s x - y\| \text{ exists}\}$.

It is the purpose of this paper that some of the weak convergence of semigroups of nonexpansive mappings carries over to the larger class of mappings defined above.

2. Opial's condition and weak convergence

Unless other specified, let $G, X, C, \mathcal{T} = \{T_s : s \in G\}$ be as before. When $\{x_\alpha\}$ is a net in X, then $x_\alpha \to x$ (resp. $x_\alpha \rightharpoonup x$) will denote norm (resp. weak) convergence of the net $\{x_\alpha\}$ to x.

We begin with the following result.

LEMMA 2.1. *For each $x \in C$, $F(\mathcal{T}) \subseteq E(x)$.*

Proof. Let $y \in F(\mathcal{T})$ and $r = \inf_{s \in G} \|T_s x - y\|$. Given $\varepsilon > 0$, there is $s_0 \in G$ such that $\|T_{s_0} x - y\| < r + \frac{\varepsilon}{2}$. Since $\mathcal{T}$ is of asymptotically nonexpansive type, there also exists $t_0 \in G$ such that

$$\|T_t T_{s_0} x - y\| \leqq \|T_{s_0} x - y\| + \frac{\varepsilon}{2}, \text{ for all } t \geqslant t_0.$$

Let $b \geqslant a_0 = t_0 s_0$. Since G is right reversible, we may assume $b \in \overline{Ga_0}$. Let $\{s_\alpha\}$ be a net in G such that $s_\alpha a_0 \to b$. Then, for each α,

$$\|T_{s_\alpha t_0} T_{s_0} x - y\| \leqq \|T_{s_0} x - y\| + \frac{\varepsilon}{2}.$$

Hence, $\|T_b x-y\| \leqq \|T_{s_0}x-y\|+\frac{\varepsilon}{2}$. So, we have

$$\inf_s \sup_{t\geq s}\|T_t x-y\| \leqq \sup_{b\geq a_0}\|T_b x-y\|$$
$$\leqq \|T_{s_0}x-y\|+\frac{\varepsilon}{2}<r+\varepsilon.$$

Since ε is arbitrary, we have

$$\inf_s \sup_{t\geq s}\|T_t x-y\| \leqq r=\inf_{s\in G}\|T_s x-y\|.$$

Therefore, $\lim_s\|T_s x-y\|$ exists and so $y\in E(x)$.

Recall that a Banach space X satisfies Opial's condition if, for $\{x_\alpha\}\subset X$, $x\in X$, $x_\alpha \rightharpoonup x$,

$$(*)\ \limsup_\alpha\|x_\alpha-x\|<\limsup_\alpha\|x_\alpha-y\|,\ \ y(\neq x)\in X$$

(see [11, Lemma 1] and [9, Lemma 2.1]). For more details, see also [4], [10].

LEMMA 2.2. *Let X satisfy Opial's condition and $x\in C$. If $\phi\neq\omega_w(x)\subseteq E(x)$, then the orbit $\mathcal{T}(x)=\{T_s x : s\in G\}$ converges weakly.*

Proof. Since $\omega_w(x)\neq\phi$, it suffices to show that $\omega_w(x)$ is a singleton. To this end, let $y_1, y_2\in\omega_w(x)$ and $y_1\neq y_2$. Then there exist subnets $\{T_{s_\alpha}x\}$, $\{T_{s_\beta}x\}$ of the net $\{T_s x : s\in G\}$ such that $T_{s_\alpha}x\rightharpoonup y_1$ and $T_{s_\beta}x\rightharpoonup y_2$, respectively. Since $y_1, y_2\in E(x)$, there also exist $d_1, d_2\geqq 0$ for which

$$d_1=\lim_s\|T_s x-y_1\|,\ \ d_2=\lim_s\|T_s x-y_2\|.$$

Then, Opial's condition $(*)$ implies that

$$d_1=\lim_\alpha\|T_{s_\alpha}x-y_1\|<\lim_\alpha\|T_{s_\alpha}x-y_2\|=d_2$$
$$=\lim_\beta\|T_{s_\beta}x-y_2\|<\lim_\beta\|T_{s_\beta}x-y_1\|=d_1,$$

which gives a contradiction. This completes the proof.

By using Lemma 2.1 and Lemma 2.2, we now obtain the following weak convergence of $\{T_s x : s\in G\}$.

THEOREM 3.3. *Let X be uniformly convex and satisfy Opial's condition* *and $x \in C$. Then, the orbit $\{T_s x : s \in G\}$ converges weakly to an element of $F(\mathcal{T})$ if and only if $F(\mathcal{T}) \neq \phi$ and $T_{ts}x - T_s x \rightharpoonup 0$ for all $t \in G$.*

Proof. We need only prove the "if" part. Since $F(\mathcal{T}) \neq \phi$, by Lemma 2. 1., $\{T_s x : s \in G\}$ is bounded; hence $\omega_w(x) \neq \phi$. By Lemma 2. 1 and Lemma 2. 2, it suffices to show that $\omega_w(x) \subseteq F(\mathcal{T})$. To this end, let $y \in \omega_w(x)$; hence there is a subnet $\{T_{s_\alpha} x\}$ of the net $\{T_s x : s \in G\}$ for which $T_{s_\alpha} x \rightharpoonup y$. Since $T_{ts}x - T_s x \rightharpoonup 0$ for all $t \in G$, we have $T_{ts_\alpha} x \rightharpoonup y$ for all $t \in G$. Suppose that $y \notin F(\mathcal{T})$ and set

$$r_t = \limsup_\alpha \|T_{ts_\alpha} x - y\|.$$

With a proof as in Lemma 2. 1, we can see that $r = \inf_s r_s = \limsup_s r_s$. Since $y \notin F(\mathcal{T})$, $r > 0$ easily follows.

For any fixed $\eta > 0$, choose $\varepsilon > 0$ so small that

$$(r+\varepsilon)\left[1-\delta\left(\frac{\eta}{r+\varepsilon}\right)\right] < r,$$

where δ is the modulus of convexity of the norm. For the $\varepsilon > 0$, there is $s_0 \in G$ such that $r_{s_0} < r + \frac{\varepsilon}{2}$. Since $\mathcal{T}$ is of asymptotically nonexpansive type, there exists $t_0 \in G$ such that

$$\|T_t T_{s_0 s_\alpha} x - T_t y\| \leqq \|T_{s_0 s_\alpha} x - y\| + \frac{\varepsilon}{2},$$

for all $t \geqslant t_0$ and each α. Then we have

$$\limsup_\alpha \|T_{ts_0} T_{s_\alpha} x - T_t y\| \leqq r_{s_0} + \frac{\varepsilon}{2} < r + \varepsilon,$$

and also

$$\limsup_\alpha \|T_{ts_0} T_{s_\alpha} x - y\| < r + \varepsilon,$$

for all $t \geqslant t_0$. Since $y \notin F(\mathcal{T})$, we choose a $b \geqslant t_0$ with $T_b y \neq y$. Then, there is $s_{\alpha_0} \in G$ such that

$$\|T_{bs_0} T_{s_\alpha} x - T_b y\|, \quad \|T_{bs_0} T_{s_\alpha} x - y\| < r + \varepsilon,$$

for all $s_\alpha \geqslant s_{\alpha_0}$. By uniform convexity of X, we have

$$\|T_{bs_0}T_{s_\alpha}x-(T_by+y)/2\| \leqq (r+\varepsilon)[1-\delta(\|T_by-y\|/(r+\varepsilon)] < r$$

for all $s_\alpha \geqslant s_{\alpha_0}$. Thus, Opial's condition (*) implies that

$$r_{bs_0} < \limsup_\alpha \|T_{bs_0}T_{s_\alpha}x-(T_by+y)/2\| \leqq \sup_{s_\alpha \geq s_{\alpha_0}} \|T_{bs_0}T_{s_\alpha}x-(T_by+y)/2\| < r.$$

This contradicts $r=\inf_s r_s$. Hence the proof is complete.

Similarly, using Lemma 2. 1 and Lemma 2. 2, we get

THEOREM 2. 4. *Let X be reflexive and satisfy Opial's condition and $x \in C$. If $E(x) \neq \phi$ and $T_sx-T_{ts}x \to 0$ for all $t \in G$, then the orbit $\{T_sx : s \in G\}$ converges weakly to an element of $F(\mathcal{T})$.*

Proof. Since $E(x) \neq \phi$, the orbit $\mathcal{T}(x)=\{T_sx : s \in G\}$ is bounded. By reflexivity of X, $\omega_w(x) \neq \phi$. By Lemma 2. 1 and Lemma 2. 2, it suffices to show that $\omega_w(x) \subseteq F(\mathcal{T})$. Let $y \in \omega_w(x)$; hence there is a subnet $\{T_{s_\alpha}x\}$ of the net $\{T_sx : s \in G\}$ such that $T_{s_\alpha}x \rightharpoonup y$. Given $\varepsilon>0$, since $\mathcal{T}$ is of asymptotically nonexpansive type, there is $t_0 \in G$ such that

$$\|T_tT_{s_\alpha}x-T_ty\| \leqq \|T_{s_\alpha}x-y\|+\varepsilon,$$

for all $t \geqslant t_0$ and each α. For fixed $t \geqslant t_0$, we have

$$\|T_{s_\alpha}x-T_ty\| \leqq \|T_{s_\alpha}x-T_{ts_\alpha}x\|+\|T_tT_{s_\alpha}x-T_ty\| \leqq \|T_{s_\alpha}x-T_{ts_\alpha}x\|+\|T_{s_\alpha}x-y\|+\varepsilon,$$

for each α. Thus,

$$\limsup_\alpha \|T_{s_\alpha}x-T_ty\| \leqq \limsup_\alpha \|T_{s_\alpha}x-y\|+\varepsilon.$$

Since ε is arbitrary, Opial's condition (*) implies that $T_ty=y$, for all $t \geqslant t_0$. So, we have $y \in F(\mathcal{T})$. This completes the proof.

When T is a nonexpansive mapping of C into C and $\mathcal{T}=\{T^n : n \in N\}$, this problem is equivalent to that of weak convergence of the sequence $\{T^nx : n \in N\}$ to a fixed point of T considered by Opial in [11] and Pazy in [12]. For right reversible semigroups of nonexpansive mappings on a Hilbert space, see Theorem 2. 3 due to Lau [9]. If X is uniformly convex and $\mathcal{T}=\{T_s : s \in G\}$ is left reversible, and if there is a point $x \in C$ such that its orbit $\mathcal{T}(x)=\{T_s x : s \in G\}$ is bounded, then by the slight modification of Theorem 1 of [7], the asymptotic center $c(x)$ of the orbit $\mathcal{T}(x)$ with respect to C is in fact a common fixed point of $\mathcal{T}$. Taking $G=N$ in Theorem 2.3, we improve Emmanuele's result [2, Theorem 2].

COROLLARY 2.5. *Let X be uniformly convex and satisfy Opial's condition If $T: C \to C$ is a mapping of asymptotically nonexpansive type, and if there is a point $x \in C$ such that $\{T^n x : n \in N\}$ is bounded, then $T^{n+1}x - T^n x \to 0$ implies that $\{T^n x\}$ converges weakly to the asymptotic center $c(x)$ of $\{T^n x : n \in N\}$ with respect to C.*

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