CONTINUITY OF THE CLOSURE OF JOINT NUMERICAL RANGE(OPERATORS THAT ARE POINTS OF JOINT SPECTRAL CONTINUITY (I))

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0. Introduction

After Newburgh [12] studied the continuity of the set-valued functions, the continuity of the spectra and the numerical range has been tremendously developed by various authors [1, 3, 4, 5, 8, 9, 11]. As much of the knowledge in a single operator has been carried to the analogous situation in the case of an n-tuple of operators, it would be quite reasonable to try to study the continuity of joint spectra and joint numerical range. In this note we show the joint numerical range is continuous with respect to the uniform operator topology and there are some operators that are points of joint spectral continuity via the continuity of the joint numerical range.

1. Notation and definition

Throughout this note, H will be a separable complex Hilbert space with the scalar product $\langle , \rangle$ and associated norm $\|\cdot\|$ and $B(H)$, the Banach algebra of bounded linear operators on H. Let $T=(T_1, T_2, \cdots, T_n)$ and $S=(S_1, S_2, \cdots, S_n)$ be n-tuples of bounded linear operators on H. For any n-tuple $T=(T_1, T_2, \cdots, T_n)$ of operators,

$$\|T\|_j = \sup\{\|T_1 x\|^2 + \|T_2 x\|^2 + \cdots + \|T_n x\|^2)^{1/2} : \|x\|=1\}$$

is called the joint operator norm. The joint left spectrum $\sigma_j(T)$ of T relative to T'', where $T=(T_1, T_2, \cdots, T_n)$ is an n-tuple of mutually commuting operators on a complex Hilbert space H and T'' is the double commutant of T, is defined as the set of all n-tuples of complex numbers

$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ with $\sum_{k=1}^{n} S_k(T_k - \lambda_k) \neq I$ for all $S_1, S_2, \cdots, S_n$ in T'', where I denotes the identity operator.

The joint numerical range $W_j(T)$ of $T = (T_1, T_2, \cdots, T_n)$ is defined as the set of all n-tuples of complex numbers

$$\{(\langle T_1 x, x \rangle, \ \langle T_2 x, x \rangle, \cdots, \langle T_n x, x \rangle) : x \in H, \ \|x\| = 1\}.$$

Hence the joint spectrum is a function defined on a Banach algebra $B(H)$ whose range consists of non-empty compact subsets of the n-dimensional unitary space C^n if T_i $(i=1, 2, \cdots, n)$ are mutually commuting [2]. The joint numerical range $W_j(T)$ is a function defined on $B(H)$ whose range consists of convex subsets of the C^n.

In the case of $j=1$, the joint spectrum and the joint numerical range are the usual left spectrum $\sigma_1(T)$ and usual numerical range $W_1(T)$ of an operator respectively.

LEMMA 1.1. *Let (C^n, d) be a metric space. Define*

$$h(A, B) = \sup\{d(a, B) : a \in A\}$$
$$and \ \rho(A, B) = \max\{h(A, B), \ h(B, A)\}$$

for $(A, B) \in \Sigma \times \Sigma$, where Σ is the collection of all non-empty compact subsets of C^n equipped with the Hausdorff metric, then ρ is a metric in Σ.

DEFINITION 1.2. [10]. Let (X, e) be a metric space. If $f : X \to \Sigma$ is a function, then f is said to be upper semi-continuous (lower semi-continuous) at $x \in X$ if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $e(x_n, x) < \delta$ implies $f(x_n) \subset f(x) + (\varepsilon)$ (respectively, $f(x) \subset f(x_n) + (\varepsilon)$). For a subset A of C, we denote $A + (\varepsilon) = \{\lambda \in C \mid \text{dist}(\lambda, A) < \varepsilon\}$ for any positive number ε.

The following lemma is due to Bezak [1].

LEMMA 1.3. *For a topological space X and a metric space (C^n, d) of finite diameter, let ρ be the Hausdorff metric induced by d on Σ. If f maps X into Σ, then f is upper semi-continuous and lower semi-continuous at $x \in X$ if f is continuous at x with respect to ρ, Conversely, if f is upper semi-continuous and lower semi-continuous at $x \in X$, then f is continuous at x with respect to ρ.*

2. Main results

Since the Hausdorff metric is defined for compact sets, the appropriate function to discuss is $CL(W_j)$, the closure of W_j, not W_j.

THEOREM 2.1. *The function $CL(W_j)$ is continuous with respect to the uniform operator topology.*

Proof. If $\|T-S\|_j < \varepsilon$ and x is a unit vector where $T = (T_1, T_2, \cdots, T_n)$, $S = (S_1, S_2, \cdots, S_n)$, then $|\langle(T-S)x, x\rangle| < \varepsilon$. Since $\langle(T_1, T_2, \cdots, T_n)x, x\rangle = \langle Tx, x\rangle = \langle Sx, x\rangle + \langle(T-S)x, x\rangle = \langle(S_1, S_2, \cdots, S_n)x, x\rangle + (\langle(T_1-S_1)x, x\rangle, \langle(T_2-S_2)x, x\rangle, \cdots, \langle(T_n-S_n)x, x\rangle), W_j(T) \subset W_j(S) + (\varepsilon)$. Symmetrically, $W_j(S) \subset W_j(T) + (\varepsilon)$.

THEOREM 2.2. [6] *Let $T = (T_1, T_2, \cdots, T_n)$ be a commuting n-tuple of normal operators. Then the closure of the joint numerical range of T is the closed convex hull of its joint spectrum.*

COROLLARY 2.3. *Let $T = (T_1, T_2, \cdots, T_n)$ be a commuting n-tuple of normal operators, then the convex hull of its joint spectrum is continuous*

THEOREM 2.4. [6] *Let $T_\phi = (T_{\phi_1}, T_{\phi_2}, \cdots, T_{\phi_n})$ be an n-tuple of analytic Toeplitz operators. Then the closure of the joint numerical range of T_ϕ is the closed convex hull of its joint spectrum.*

COROLLARY 2.5. *Let $T = (T_{\phi_1}, T_{\phi_2}, \cdots, T_{\phi_n})$ be an n-tuple of analytic Toeplitz operators, then the closed convex hull of its joint spectrum is continuous.*

COROLLARY 2.6. *The function $CL(W_1)$, the closure of numerical range, is continuous.*

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