Comm. Korean Math. Soc. 2 (1987), No. 1, pp. 53~57

UNIQUENESS FOR THE LEWY OPERATOR

JONGSIK KIM AND YONG JING SUK

Let B be an open ball about the origin in R^3. We denote a variable point in R^3 by (x_1, x_2, x_3).

Let L be a linear partial differential operator of the first order defined by

$$(1) \qquad L=\frac{1}{2}\left(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\right)+i(x_1+ix_2)\frac{\partial}{\partial x_3}.$$

Hans Lewy presented this operator, now famous, showing that for most (in suitable sense) C^∞ function f the equation

$$Lu=f$$

admits no solution in any open set in R^3. Shortly thereafter Hörmander derived a general necessary condition for local solvability of any linear partial differential operator of arbitrary order.

In this paper we prove that the C^1 solution satisfying

$$Lh=0 \text{ in } B,$$
$$h|_{x_3<0}=0$$

must vanish in a full neighborhood of an arbitrary point $(x_1, x_2, 0)$ in B.

This result can be obtained by the Holmgren's uniqueness theorem as the (x_1, x_2) plane is a noncharacteristic hypersurface with respect to the operator L (cf. Remark 1).

Here, however, we offer another proof based on a technique of the local constancy principle. Such an approach will be helpful to develop the method of uniqueness proof for another operators with nonanalytic coefficients.

We notice that the homogeneous equation

This research is supported by the KOSEF research grant 1987.

$$Lz=0$$

has a solution

$$z=x_1+ix_2.$$

We notice that

$$x_1=\frac{z+\bar{z}}{2} \quad \text{and} \quad x_2=\frac{z-\bar{z}}{2i},$$

from which it follows that x_1, x_2 are C^1 functions z and $\bar{z}$. Now let h be a C^1 function satisfying

$$Lh=0 \text{ in } B,$$
$$h|_{x_3<0}=0.$$

We set

$$h(x_1, x_2, x_3)=\tilde{h}(z, \bar{z}, x_3).$$

Then $\tilde{h}$ is a C^1 function of $z, \bar{z}$ and x_3. We have

(2)
$$\begin{aligned} 0&=Lh \\ &=\frac{\partial\tilde{h}}{\partial z}Lz+\frac{\partial\tilde{h}}{\partial\bar{z}}L\bar{z}+\frac{\partial\tilde{h}}{\partial x_3}Lx_3 \\ &=\frac{\partial\tilde{h}}{\partial\bar{z}}L\bar{z}+\frac{\partial\tilde{h}}{\partial x^3}Lx_3 \end{aligned}$$

as $Lz=0$. Moreover,

$$\begin{aligned} L\bar{z}&=\left[\frac{1}{2}\left(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\right)+i(x_1+ix_2)\frac{\partial}{\partial x_3}\right](x_1-ix_2) \\ &=1, \end{aligned}$$

$$\begin{aligned} Lx_3&=\left[\frac{1}{2}\left(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\right)+i(x_1+ix_2)\frac{\partial}{\partial x_3}\right](x_3) \\ &=i(x_1+ix_2) \\ &=iz. \end{aligned}$$

Therefore (2) reads as

(3)
$$\frac{\partial\tilde{h}}{\partial\bar{z}}+iz\frac{\partial\tilde{h}}{\partial x_3}=0.$$

Now we set

$$z=x_1+ix_2=re^{i\theta}$$

so that

$$\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\right)=\frac{e^{i\theta}}{2}\left(\frac{\partial}{\partial r}+\frac{i}{r}\ \frac{\partial}{\partial \theta}\right).$$

Then (3) reads as

$$\frac{1}{2}\ \frac{\partial \tilde{h}}{\partial r}+\frac{i}{2r}\ \frac{\partial \tilde{h}}{\partial \theta}+ir\frac{\partial \tilde{h}}{\partial x_3}=0; \tag{4}$$

that is,

$$\frac{1}{2r}\ \frac{\partial \tilde{h}}{\partial r}+\frac{i}{2r^2}\ \frac{\partial \tilde{h}}{\partial \theta}+i\frac{\partial \tilde{h}}{\partial x_3}=0,$$

where we view $\tilde{h}$ as a function in (r, θ, x_3).

Fixing $\theta=\theta_0$, we get

$$\frac{1}{2r}\ \frac{\partial \tilde{h}}{\partial r}+i\frac{\partial \tilde{h}}{\partial x_3}=0; \tag{5}$$

that is,

$$\left(\frac{\partial}{\partial \rho}+i\frac{\partial}{\partial x_3}\right)\tilde{h}=0,$$

where we set $r^2=\rho$ and set again $\tilde{h}$ as a function in (ρ, x_3).

The projection $\pi(B)$ of B onto (ρ, x_3) plane is open and connected and $\tilde{h}$ is a holomorphic function of $\rho+ix_3$ in $\pi(B)$ for $\rho>0$.

We notice that $\tilde{h}$ vanishes for $x_3<0$ (in particular, on the negative x_3-axis, where $\rho=0$).

It follows that $\tilde{h}$ must vanish in $\pi(B)$.

Consequently, the function $h(x_1, x_2, x_3)$ must vanish in a full neighborhood $\pi(B)\backslash x_3$-axis of an arbitrary point $(x_1, x_2, 0)$ in B except the origin. But h is a C^1 function in B, and hence we may include the origin.

Thus we can summarize our arguments as follows.

THEOREM 1. *Let* (x_1, x_2, x_3) *be the point in* R^3 *and* L *be the Lewy operator defined by* (3. 1). *If* h *is a* C^1 *solution of* $Lh=0$ *in an open ball* B *about the origin in* R^3 *and*

$$h|_{x_3<0}=0,$$

then h must vanish in a full neighborhood of an arbitrary point $(x_1, x_2, 0)$ *in B.*

REMARK 1. $$\begin{aligned} L&=\frac{1}{2}\left(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\right)+i(x_1+ix_2)\frac{\partial}{\partial x_3} \\ &=-\frac{1}{2}\left(\frac{1}{i}\ \frac{\partial}{\partial x_2}\right)-(x_1+ix_2)\left(\frac{1}{i}\ \frac{\partial}{\partial x_3}\right) \\ &\quad+\frac{i}{2}\left(\frac{1}{i}\ \frac{\partial}{\partial x_1}\right) \\ &=\frac{i}{2}D_{x_1}-\frac{1}{2}D_{x_2}-(x_1+ix_2)D_{x_3}, \end{aligned}$$

where D_{x_j} stands, as usual, for $\frac{1}{i}\ \frac{\partial}{\partial x_j}$.

We notice that

$$\begin{aligned} L(0,0,1)&=\frac{i}{2}\cdot 0-\frac{1}{2}\cdot 0-(x_1+ix_2)\cdot 1 \\ &=-(x_1+ix_2)\neq 0 \end{aligned}$$

if $(x_1, x_2)\neq 0$.

Also notice that L has analytic coefficients.

By the Holmgren's uniqueness theorem, C^1 solution h satisfying

$$\begin{aligned} &Lh=0 \text{ in } B, \\ &h|_{x_3=0}=0 \end{aligned}$$

must vanish in a neighborhood of an arbitrary point $(x_1, x_2, 0)$ in B.

REMARK 2. $L=\frac{i}{2}D_{x_1}-\frac{1}{2}D_{x_2}-(x_1+ix_2)D_{x_3}$.

Its principal symbol is

$$p(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3)=\left(-\frac{1}{2}\xi_2-x_1\xi_3\right)+i\left(\frac{1}{2}\xi_1-x_2\xi_3\right).$$

At the point $x_1=x_2=0,\ \xi_2=0,\ \xi_1=0,\ \xi_3\neq 0$, we get

$$\operatorname{Re} p=0,\ \operatorname{Im} p=0.$$

But we have

$$\{\mathrm{Re}\ p,\ \mathrm{Im}\ p\} = 0\cdot 0 - (-\xi_3)\cdot\frac{1}{2} + \left(-\frac{1}{2}\right)\cdot(-\xi_3) - 0\cdot 0 + (-x_1)\cdot 0 - 0\cdot(-x_2)$$

$$= \frac{1}{2}\xi_3 + \frac{1}{2}\xi_3$$
$$= \xi_3 \neq 0$$

as $\xi_3 \neq 0$.

So by the result of S. Alinhac (cf. [1]) there exist C^∞ functions $u(x_1, x_2, x_3)$ in R^3 and $a(x_1, x_2, x_3)$ with support in $\{(x_1, x_2, x_3)\,|\,x_3 \geq 0\}$ such that

$$Lu - au = 0,$$
$$\text{origin} \in \text{supp } u.$$

References

1. S. Alinhac, *Non unicitè de problème de Cauchy pour des operateurs de type principal*, Seminaire Goulaouic-Schwartz 16, Ecole Polytech. Paris (1981).
2. H. Jacobowitz and F. Treves, *Nonrealizable CR structures*, Invent. Math. **66**, 231-249 (1982).
3. L. Nirenberg, *Lecture on linear partial differential equations*, Reg. Confer. Series in Math. No. 17, (1973), Amer. Math. Soc.
4. F. Treves, *Remarks about certain first order linear partial differential equations in two variables*, Comm. in PDE, 5, 381-425 (1980).

Seoul National University
Seoul 151, Korea.
and
Kangweon National University
Chuncheon 200, Korea