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A NOTE ON THE SPACE $H^{p,a}$

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1. Introduction

For $0<p<\infty$ and $0<a\leq 1$, $H^{p,a}$ is defined to be the space of those $f\in H^p$ for which

$$\|f\|_{p,a}=\max(\|f\|_p, \sup_{|z|\leq 1}(1-|z|)^{a/p}|f(z)|)<\infty,$$

where H^p denotes the Hardy space on the unit disc of the complex plane with its norm $\| \ \|_p$. See [8] and [9] for the previous results on $H^{p,a}$.

In this note, we settle the following conjectures of ours in [9].

CONJECTURE 1. If $f\in H^{p,a}$, $0<p<q<\infty$, and $\lambda\geq p$, then

$$(1) \qquad \int_0^1 (1-r)^{\lambda a(1/p-1/q)-1} M_q(r,f)^\lambda dr \leq C_{p,q,a}\|f\|^\lambda_{p,a},$$

where $C_{p,q,a}$ is a positive constant not depending on f and $M_q(r,f)$ is the usual q-integral-mean of $|f(z)|$ on $|z|=r$.

CONJECTURE 2. If $f(z)=\sum_0^\infty a_n z^n \in H^{p,a}(0<p<1)$, then

$$(2) \qquad \sum_0^\infty (n+1)^{ap(1-1/p)-1}|a_n|^p<\infty.$$

Since $H^{p,1}=H^p$, (1) reduces to the Hardy-Littlewood imbedding theorem [5, Theorem 5.11] when $a=1$. In [9], we proved a weak type inequality of an operator which supports the truth of Conjecture 1. Unfortunately, Conjecture 1 turns out to be false as we shall see in Theorem 2.

Conjecture 2 is a consequence of Conjecture 1. In spite of the failure

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of the latter, we shall see in the corollary of Theorem 1 that the former remains true.

2. Taylor coefficients

For $0<p, q<\infty$, and $\alpha>-1$, $A^{p,q,\alpha}$ and $l(p,q)$ are defined respectively to be the space of those f analytic in the unit disc satisfying

$$\int_0^1 (1-r)^\alpha M_q(r,f)^p dr<\infty,$$

and the space of those sequences $\{b_n\}_0^\infty$ satisfying

$$\sum_{n=0}^{\infty}\Big(\sum_{k\in I_n}|b_k|^p\Big)^{q/p}<\infty,$$

where $I_n=\{k: 2^{n-1}\leq k<2^n\}$ $(n=1,2,\cdots)$ and $I_0=\{0\}$. When $p=q$, we denote $A^{p,p,\alpha}$ by $A^{p,\alpha}$. $l(\infty,q)$ is defined similarly by replacing p-mean by a supremum. See [4] and [7] for these spaces.

If $f(z)=\sum_0^\infty b_n z^n$ is analytic in the unit disc and γ real, identifying f as the sequence of its Taylor coefficients, we denote $\{(n+1)^{-\gamma}a_n\}_0^\infty$ by $I^\gamma f$, the fractional integral of order γ [6]. Also, for a set S of sequences, we denote $\{I^\gamma f: f\in S\}$ by $I^\gamma S$.

THEOREM A. [8, Theorem B] *If* $0<p<q<\infty$, *then*

(3) $$H^{p,\alpha}\subset A^{q,-1+\alpha(q-p)/p}.$$

THEOREM B. [10, Theorem 1] *For* $0<p<\infty$, $1\leq q\leq 2$, $\alpha>-1$, *and* $1/q+1/q'=1$,

(4) $$A^{p,q,\alpha}\subset I^{-(1+\alpha)/p}l(q', \max(1,p)).$$

In fact, the proof of Theorem 1 in [10] says a little more:

(4′) $$A^{p,q,\alpha}\subset I^{-(1+\alpha)/p}l(q',p),\quad 1\leq q<2,$$

(4″) $$A^{p,2,\alpha}=I^{-(1+\alpha)/p}l(2,p),\quad q=2.$$

The conjecture 1 is a corollary of the following stronger theorem.

THEOREM 1. *If* $0<p\leq 2$ *and* $0<a\leq 1$, *then*

$$H^{p,a}\subset I^{a(1/2-1/p)}l^2. \tag{5}$$

Proof. The case $a=1$ of (5) is known (See [10, Corollary 1]). Also, the case $p=2$ is obvious. If $0<p<2$, and $0<a<1$, by taking $q=2$ in (3) and (4) we have

$$\begin{aligned} H^{p,a} &\subset A^{2,-1+a(2-p)/p} \quad \text{(by (3))} \\ &\subset I^{a(1/2-1/p)}l^2 \quad \text{(by (4))}, \end{aligned}$$

which proves (5).

We will show in section 4 that the containment (5) is sharp in a certain sense for the case $1/2<a\leq p<1$.

COROLLARY. *Conjecture 2 is true.*

Proof. The assertion of Conjecture 2 is equivalent to

$$H^{p,a}\subset I^{a(1-1/p)-1/p}l^p, \quad \text{for } 0<p<1,\ 0<a\leq 1.$$

If $a=1$, it is a theorem of Hardy and Littlewood [5, Theorem 6. 2]. Suppose $0<a<1$. By (5), it is enough to prove that

$$I^{a(1/2-1/p)}l^2\subset I^{a(1-1/p)-1/p}l^p,$$

which is equivalent to

$$I^{1/p-a/2}l^2\subset l^p.$$

This containment is true by Hölder's inequality.

3. Derivative of a Blaschke product

We give an example of a Blaschke product whose derivative disproves conjecture 1. If $B(z)$ is a Blaschke product formed by $\{z_n\}$, we denote $1-|z_n|$ by d_n as usual. See [1], [2], [3], [8], and [11] for derivatives of inner functions.

THEOREM C. [1] *If f is inner and* $1/2<p<1$, *then* $f'\in A^{1,-p}$ *if and only if* $f'\in H^p$.

THEOREM D. [11, Theorem 5 (b)] *Suppose that* $1\leq q<\infty$, $0<p<\infty$, *and* $1/2<\alpha q<1$. *If* $B(z)$ *is a Blaschke product whose zeros* $\{z_n\}$ *tend to* 1 *nontangentially, then*

$$B'\in A^{p,q,-1+p(1-\alpha)}$$

if and only if

$$\{d_n^{1/q-\alpha}n^{\alpha-1/p}\}_1^\infty\in l^p.$$

By [2, Theorem 3] and [3, Theorem 3], the f in Theorem C is in fact a Blaschke product.

The next theorem shows that the conjecture 1 is false in the case $\frac{1}{2}\leq p=\alpha<1$ and $1\leq q<\infty$.

THEOREM 2. *If* $\frac{1}{2}<p<1$ *and* $1\leq q<\infty$, *then there is a Blaschke product* $B(z)$ *such that*

$$B'\in H^{p,p} \text{ but } B'\notin A^{p,q,-1+p(1-p/q)}.$$

Proof. Fix such p and q. Let

$$d_n=[n^{-p}(\log n)^{-q/p}]^{1/(1-p)}$$

and $z_n=1-d_n$, $n=2,3,\cdots$. Then $\sum d_n<\infty$. Let $B(z)$ be the Blaschke product formed by $\{z_n\}$. Since

$$\sum_2^\infty (d_n/n)^{1-p}=\sum_2^\infty n^{-1}(\log n)^{-q/p}<\infty,$$

$B'\in A^{1,-p}$ by Theorem D, and so $B'\in H^p$ by Theorem C. Thus, $B'\in H^{p,p}$ because of the fact that $B'(z)=O((1-|z|)^{-1})$.

On the other hand, it is easy to see that

$$\{d_n^{(1-p)/q}n^{p/q-1/p}\}\notin l^p;$$

so that $B'\notin A^{p,q,-1+p(1-p/q)}$ by Theorem D again.

4. Remark

As an application of Theorem 2, we show that Theorem 1 is sharp when $\frac{1}{2}<\alpha\leq p<1$ in the sense that if $s<2$ then there exists an

$$f \in H^{p,a} \setminus I^{a(1/2-1/p)} l(2, s).$$

First suppose that $1/2 < a = p < 1$. By Theorem 2, there is a Blaschke product B such that

$$B' \in H^{a,a} \setminus A^{s,2,-1+s(1-a/2)}.$$

Therefore we have by (4″)

$$B' \in H^{a,a} \setminus I^{a/2-1} l(2, s). \tag{6}$$

For the case $1/2 < a < p < 1$, we need

THEOREM E. [8, Theorem 2.1] *If $f \in H^p$ and $f(z) = O((1-|z|)^{-\gamma})$ with $0 < \gamma \leq 1/p$, then $I^{\beta} f \in H^q$ with $q = \gamma p/(\gamma - \beta)$ where $0 < \beta < \gamma$.*

Now let $f(z) = I^{1-a/p} B'(z)$, where $B(z)$ is a Blaschke product satisfying (6). Then $f \in H^{p,a}$ by Theorem E. It follows from the fact $B' \notin I^{a/2-1} l(2, s)$ that $f \notin I^{a(1/2-1/p)} l(2, s)$.

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