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REMARKS ON CONVOLUTION OF UNIVALENT FUNCTIONS

Yong Chan Kim and Oh Sang Kwon

1. Introduction

Let S denote the class of functions of the form $f(z)=z+\sum_2^\infty a_n z^n$ that are analytic and univalent in the unit disk U. A function $f\in S$ is said to be starlike of order α, $0\leq\alpha<1$, denote by $S(\alpha)$, if $\mathrm{Re}(zf'(z)/f(z))>\alpha$ for z in U and is said to be convex of order α, $0\leq\alpha<1$, denoted by $K(\alpha)$, if $\mathrm{Re}\{1+(zf''(z)/f'(z))\}>\alpha$ for z in U.

Let T be the class of functions in S of the form $f(z)=z-\sum_2^\infty |a_n| z^n$ and set $S^*(\alpha)=s(\alpha)\cap T$ and $K^*(\alpha)=K(\alpha)\cap T$. Also, let $p^*(\alpha)$ denote the class of functions $f(z)=z-\sum_2^\infty |a_n|z^n$ analytic in U satisfying $\mathrm{Re}\ f'(z)>\alpha$ for z in U.

If $f(z)=z+\sum_2^\infty a_n z^n$ and $g(z)=z+\sum_2^\infty b_n z^n$ are analytic in U, then so is their Hadamard product or convolution denoted by $(f*g)(z)$, and defined by $(f*g)(z)=z+\sum_2^\infty a_n b_n z^n$.

A function $f\in S$ is said to be pre-starlike of order α, $0\leq\alpha<1$, denoted by $R(\alpha)$, if $f(z)*z(1-z)^{2\alpha-2}\in S(\alpha)$ and set $R^*(\alpha)=R(\alpha)\cap T$.

In [1], we investigated the mapping properties of the function $F(z)$ denoted by

$$F(z)=(1-\lambda)(f*g)+\lambda(f*h) \tag{1.1}$$

where $g(z)=z+\sum_2^\infty b_n z^n$, $h(z)=z+\sum_2^\infty c_n z^n$ (b_n and c_n are known and non-negative), $\lambda\geq 0$ and when $f(z)$ is respectively in T, $S^*(\alpha)$, $K^*(\alpha)$,

$P^*(\alpha)$ or $R^*(\alpha)$.

In the present paper, we study the Distortion Theorem and Fractional Calculus of $F(z)$ when $f(z)$ belongs respectively to several subclasses of analytic functions with negative coefficients.

From now on, the function $F(z)$ will be defined by $F(z)=(1-\lambda)(f*g)+\lambda(f*h)$ with $g(z)=z+\sum_2^\infty b_n z^n$, $h(z)=z+\sum_2^\infty c_n z^n$ in which λ, b_n and c_n are real and nonnegative.

2. Distortion Theorems

THEOREM 2.1. *Let* $f(z)\in T$ *and* $A(n)\geq A(n+1)\geq 0$ *for* $n\geq 2$, *where* $A(n)=b_n-\lambda b_n+\lambda c_n$. *Then*

(2.1) $$|z|-\frac{b_2-\lambda b_2+\lambda c_2}{2}|z|^2\leq |F(z)|\leq |z|+\frac{b_2-\lambda b_2+\lambda c_2}{2}|z|^2$$
for $z\in U$.

(2.2) $$1-(b_2-\lambda b_2+\lambda c_2)|z|\leq |F'(z)|\leq 1+(b_2-\lambda b_2+\lambda c_2)|z|$$
for $z\in U$.

The result is sharp.

Proof. We note that

$$2\sum_2^\infty |a_n|\leq \sum_2^\infty n|a_n|\leq 1$$

by ([5], Theorem 3). Therefore we have

$$\begin{aligned}|F(z)| &\geq |z|-|z|^2\sum_2^\infty (b_n-\lambda b_n+\lambda c_n)|a_n|\\ &\geq |z|-(b_2-\lambda b_2+\lambda c_2)|z|^2\sum_2^\infty |a_n|\\ &\geq |z|-\frac{b_2-\lambda b_2+\lambda c_n}{2}|z|^2\end{aligned}$$

and

$$\begin{aligned}|F(z)| &\leq |z|+|z|^2\sum_2^\infty (b_n-\lambda b_n+\lambda c_n)|a_n|\\ &\leq |z|+(b_2-\lambda b_2+\lambda c_2)|z|^2\sum_2^\infty |a_n|\end{aligned}$$

$$\leq |z| + \frac{b_2 - \lambda b_2 + \lambda c_2}{2} |z|^2$$

Using ([5], Theorem 3), we have

$$|F'(z)| \geq 1 - (b_2 - \lambda b_2 + \lambda c_2) |z| \sum_2^{\infty} n |a_n|$$
$$\geq 1 - (b_2 - \lambda b_2 + \lambda c_2) |z|.$$

and

$$|F'(z)| \leq 1 + (b_2 - \lambda b_2 + \lambda c_2) |z| \sum_2^{\infty} n |a_n|$$
$$\leq 1 + (b_2 - \lambda b_2 + \lambda c_2) |z|.$$

The result is sharp for the function $f(z) = z - (1/2) z^2$.

REMARK. Under the hypotheses of Theorem 2.1 and $z \in U$, we have the following results.

(1) If $f(z) \in S^*(\alpha)$, then

(i) $|z| - \dfrac{(b_2 - \lambda b_2 + \lambda c_2)(1-\alpha)}{2-\alpha} |z|^2 \leq |F(z)|$

$\leq |z| + \dfrac{(b_2 - \lambda b_2 + \lambda c_2)(1-\alpha)}{2-\alpha} |z|^2,$

(ii) $1 - \dfrac{2(b_2 - \lambda b_2 + \lambda c_2)(1-\alpha)}{2-\alpha} |z| \leq |F'(z)|$

$\leq 1 + \dfrac{2(b_2 - \lambda b_2 + \lambda c_2)(1-\alpha)}{2-\alpha} |z|.$

(2) If $f(z) \in K^*(\alpha)$, then

(i) $|z| - \dfrac{(b_2 - \lambda b_2 + \lambda c_2)(1-\alpha)}{2(2-\alpha)} |z|^2 \leq |F(z)|$

$\leq |z| + \dfrac{(b_2 - \lambda b_2 + \lambda c_2)(1-\alpha)}{2(2-\alpha)} |z|^2,$

(ii) $1 - \dfrac{(b_2 - \lambda b_2 + \lambda c_2)(1-\alpha)}{2-\alpha} |z| \leq |F'(z)|$

$\leq 1 + \dfrac{(b_2 - \lambda b_2 + \lambda c_2)(1-\alpha)}{2-\alpha} |z|.$

(3) If $f(z) \in P^*(\alpha)$, then

(i) $|z| - \dfrac{(b_2 - \lambda b_2 + \lambda c_2)(1-\alpha)}{2} |z|^2 \leq |F(z)|$

$$\leq |z| + \frac{(b_2 - \lambda b_2 + \lambda c_2)(1-\alpha)}{2} |z|^2,$$

(ii) $1 - (b_2 - \lambda b_2 + \lambda c_2)(1-\alpha)|z| \leq |F'(z)|$
$\leq 1 + (b_2 - \lambda b_2 + \lambda c_2)(1-\alpha)|z|.$

THEOREM 2.2. *Let $f(z) \in R^*(\alpha)$ with $0 \leq \alpha \leq (1/2)$ and $A(n) \geq A(n+1) \geq 0$ for $n \geq 2$, where $A(n) = b_n - \lambda b_n + \lambda c_n$. Then*

(2.3) $|z| - \dfrac{b_2 - \lambda b_2 + \lambda c_2}{2(2-\alpha)} |z|^2 \leq |F(z)| \leq |z| + \dfrac{b_2 - \lambda b_2 + \lambda c_2}{2(2-\alpha)} |z|^2$
for $z \in U$.

(2.4) $1 - \dfrac{b_2 - \lambda b_2 + \lambda c_2}{2-\alpha} |z| \leq |F'(z)| \leq 1 + \dfrac{b_2 - \lambda b_2 + \lambda c_2}{2-\alpha} |z|$
for $z \in U$.

The result is sharp for the function $f(z) = z - \{1/2(2-\alpha)\} z^2$.

Proof. (2.3) In [6], $z(1-z)^{2\alpha-2} = z + \sum_2^\infty B(\alpha, n) z^n$ where $B(\alpha, n) = \{\prod_2^n (k - 2\alpha)\} / (n-1)!\}$, $n \geq 2$.

Since $B(\alpha, n) \leq B(\alpha, n+1)$ for the $0 \leq \alpha \leq (1/2)$, It follows from [6] that

$$(2-\alpha) B(\alpha, 2) \sum_2^\infty |a_n| \leq \sum_2^\infty (n-\alpha) B(\alpha, n) |a_n| \leq 1 - \alpha.$$

Thus

$$\sum_2^\infty |a_n| \leq 1/\{2(2-\alpha)\}. \tag{2.5}$$

The inequalities in (2.3) follow from (2.5).

Using $\dfrac{(2-\alpha) B(\alpha, 2)}{2} \sum_2^\infty n |a_n| \leq \sum_2^\infty (n-\alpha) B(\alpha, n) |a_n| \leq 1 - \alpha$, we have the inequalities in (2.4).

3. Fractional Calculus

We need the following definitions of fractional derivatives and fractional integrals which were defined by Owa ([3], [4]).

DEFINITION 3.1. The fractional integrals of order δ is defined by

$$D_z^{-\delta}f(z)=\{1/\Gamma(\delta)\}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{1-\delta}}d\zeta, \tag{3.1}$$

where $\delta>0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origion and the multiplicity of $(z-\zeta)^{\delta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

DEFINITION 3.2. The fractional derivative of order δ is

$$D_z^{\delta}f(z)=\{1/\Gamma(1-\delta)\}\frac{d}{dz}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{\delta}}d\zeta, \tag{3.2}$$

where $0\leq\delta<1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origion and the multiplicity of $(z-\zeta)^{-\delta}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

DEFINITION 3.3. Under the hypotheses of Definition 3.2, the fractional derivative of order $n+\delta$ is defined by

$$D_z^{n+\delta}f(z)=\frac{d^n}{dz^n}D_z^{\delta}f(z), \tag{3.3}$$

where $0\leq\delta<1$ and $n\in N_0=\{0,1,2,3,\cdots\}$.

THEOREM 3.4. *Let $f(z)\in T$ and $A(n)\geq A(n+1)\geq0$ for $n\geq2$, where $A(n)=b_n-\lambda b_n+\lambda c_n$. Then*

$$|D_z^{-\delta}F(z)|\geq\frac{|z|^{1+\delta}}{\Gamma(2+\delta)}\left\{1-\frac{b_2-\lambda b_2+\lambda c_2}{2+\delta}|z|\right\} \tag{3.4}$$

$$|D_z^{-\delta}F(z)|\leq\frac{|z|^{1+\delta}}{\Gamma(2+\delta)}\left\{1+\frac{b_2-\lambda b_2+\lambda c_2}{2+\delta}|z|\right\} \tag{3.5}$$

for $\delta>0$ and $z\in U$. The equalities in (3.4) *and* (3.5) *are attained for the function $f(z)=z-(1/2)z^2$.*

Proof. Let

$$\begin{aligned}G(z)&=\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}F(z)\\&=z-\sum_2^{\infty}\frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)}(b_n-\lambda b_n+\lambda c_n)a_nz^n\\&=z-\sum_2^{\infty}\phi(n)(b_n-\lambda b_n+\lambda c_n)a_nz^n,\end{aligned} \tag{3.6}$$

where $\phi(n)=\{\Gamma(n+1)\Gamma(2+\delta)\}/\Gamma(n+1+\delta)\quad(n\geq 2)$.

Since we have $0<\phi(n)\leq\phi(2)=\{\Gamma(3)\Gamma(2+\delta)\}/\Gamma(3+\delta)=2/(2+\delta)$, and $f(z)\in T$, we have

$$|G(z)|\geq|z|-(b_2-\lambda b_2+\lambda c_2)\phi(2)|z|^2\sum_2^\infty|a_n| \tag{3.6}$$
$$\geq|z|-\frac{b_2-\lambda b_2+\lambda c_2}{2+\delta}|z|^2$$

and

$$|G(z)|\leq|z|+(b_2-\lambda b_2+\lambda c_2)\phi(2)|z|^2\sum_2^\infty|a_n| \tag{3.7}$$
$$\leq|z|+\frac{b_2-\lambda b_2+\lambda c_2}{2+\delta}|z|^2.$$

Thus we have the inequalities (3.4) and (3.5) from (3.6) and (3.7).

Finally, we note that the equalities in (3.4) and (3.5) are attained for the function $F(z)$ defined by

$$D_z^{-\delta}F(z)=\frac{z^{1+\delta}}{\Gamma(2+\delta)}\left\{1-\frac{b_2-\lambda b_2+\lambda c_2}{2+\delta}z\right\}$$

which is equivalent to

$$f(z)=z-(1/2)z^2.$$

This complete the proof of Theorem 3.4.

THEOREM 3.5. *Let $f(z)\in T$ and $A(n)\geq A(n+1)\geq 0$ for $n\geq 2$, where $A(n)=b_n-\lambda b_n+\lambda c_n$. Then*

$$|D_z^\delta F(z)|\geq\frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left\{1-\frac{b_2-\lambda b_2+\lambda c_2}{2-\delta}|z|\right\} \tag{3.8}$$

and

$$|D_z^\delta F(z)|\leq\frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left\{1+\frac{b_2-\lambda b_2+\lambda c_2}{2-\delta}|z|\right\} \tag{3.9}$$

for $0\leq\delta<1$ and $z\in U$. The equalities in (3.8) and (3.9) are attained for the function $f(z)=z-(1/2)z^2$.

Proof. We define $H(z)$ by

$$H(z)=\Gamma(2-\delta)z^{\delta}D_z^{\delta}F(z)$$
$$=z-\sum_2^{\infty}\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}(b_n-\lambda b_n+\lambda c_n)a_nz^n$$
$$=z-\sum_2^{\infty}\phi(n)(b_n-\lambda b_n+\lambda c_n)na_nz^n,$$

where $\phi(n)=\{\Gamma(n)\Gamma(2-\delta)\}/\Gamma(n+1-\delta)\ (n\geq2)$.

Since $0<\phi(n)\leq\phi(2)=\{\Gamma(2)\Gamma(2-\delta)\}/\Gamma(3-\delta)=1/(2-\delta)$ and $f(z)\in T$, we obtain

$$|H(z)|\geq|z|-\phi(2)(b_2-\lambda b_2+\lambda c_2)|z|^2\sum_2^{\infty}n|a_n|.$$
$$\geq|z|-\frac{b_2-\lambda b_2+\lambda c_2}{2-\delta}|z|^2$$

and

$$|H(z)|\leq|z|+\phi(2)(b_2-\lambda b_2+\lambda c_2)|z|^2\sum_2^{\infty}n|a_n|.$$
$$\leq|z|+\frac{b_2-\lambda b_2+\lambda c_2}{2-\delta}|z|^2.$$

Futher, the equalities in (3. 8) and (3. 9) are attained for the function $F(z)$ given by

$$D_z^{\delta}F(z)=\frac{z^{1-\delta}}{\Gamma(2-\delta)}\left\{1-\frac{b_2-\lambda b_2+\lambda c_2}{2-\delta}z\right\}$$

which is equivalent to

$$f(z)=z-(1/2)z^2.$$

REMARK. In the other case, it can be proved by similar method given in Theorem 3. 4 and Theorem 3. 5.

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Yeungnam University
Gyongsan 632, Korea
and
Pusan Sanub University
Pusan 608, Korea