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HARDY-LITTLEWOOD INEQUALITIES FOR THE WEIGHTED BERGMAN SPACES

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1. Preliminaries

For $z=(z_1, \cdots, z_n)$ and $\zeta=(\zeta_1, \cdots, \zeta_n)\in \boldsymbol{C}^n$, we write $\langle z, \zeta\rangle = z_1\cdot\bar{\zeta}_1+\cdots+z_n\cdot\bar{\zeta}_n$, $\|z\|=\langle z, z\rangle^{1/2}$. $B=\{z\in\boldsymbol{C}^n : \|z\|<1\}$ is the unit ball in $\boldsymbol{C}^n$ and Δ is the unit disk in $\boldsymbol{C}$. The class of all holomorphic functions on a domain D in $\boldsymbol{C}^n$ is denoted by $O(D)$. For $q>0$ we define

$$dv_q(z)=\frac{\Gamma(n+q)}{\pi^n\Gamma(q)}(1-\|z\|^2)^{q-1}dv(z)$$

where dv is the Euclidean volume element. If we let $dv_0(z)$ mean the unit surface measure $d\sigma$ on the boundary ∂B of B, then we see that $\int dv_q(z)=1 (q\geq 0)$.

With these notations the weighted Bergman spaces are defined in an obvious manner:

$$A_q^p(B)=A_q^p=\{f\in O(B) : \|f\|_{p,q}<\infty\}.$$

where $\|f\|_{p,q}=\left\{\int |f(z)|^p dv_q\right\}^{1/p}$ for $q>0$ and $\|f\|_{p,0}=\sup M_p(f:r)=\sup \left\{\int |f(rz)|^p dv_0(z)\right\}^{1/p}$. The following theorems which are the starting point of this research is due to Beatrous and Burbea [1].

THEOREM 1.1. *For* $0<p<r<\infty$, $p\leq k<\infty$, $q\geq 0$ *and* $f\in A_q^p$, *we have*

$$\left\{\int_0^1 (1-\rho)^{k\beta-1}M_r^k(f:\rho)\,d\rho\right\}^{1/k}\leq C_p\|f\|_{p,q}$$

where $\beta=(n+q)/p-n/r$ *and* C_p *is independent of* f.

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THEOREM 1.2. *For* $0<p_2\leq p_1<\infty$ *and* q_1, $q_2\geq 0$ *with* $(n+q_1)/p=(n+q_2)/p_2$, *there is a continuous injection from* $A_{q_2}^{p_2}$ *into* $A_{q_1}^{p_1}$.

2. Estimates of Taylor coefficients of functions in A_q^p

For $\alpha=(\alpha_1,\cdots,\alpha_n)\in Z_+^n$ and $z=(z_1,\cdots,z_n)\in C^n$ we write $|\alpha|=\alpha_1+\cdots\alpha_n$, $\alpha!=\alpha_1!\cdots\alpha_n!$ and $z^\alpha=z_1^{\alpha_1}\cdots z_n^{\alpha_n}$.

We prove the following generalization of a Hardy-Littlewood inequality [6].

THEOREM 2.1. *Let* $f(z)=\sum_{\alpha\geq 0}a_\alpha z^\alpha$ *be in* A_q^p. *Then for* $0<p\leq 2$ *and* $q\geq 0$, *we have*

$$(2.1)\quad \sum_{\alpha\geq 0}(|\alpha|+1)^{(n+q/2)(p-2)}\left(\frac{\alpha!}{\Gamma(n+|\alpha|+q)}\right)^{p/2}|a_\alpha|^p\leq c_p\|f\|_{p,q}^p$$

where c_p *is independent of* f *and the exponent is best possible when* $n=1$.

For the proof, we derive some estimates $M_1\ (f:\rho)$ of and follow the idea of T.M. Flett [5].

LEMMA 2.2. *If* $a_i>0$, $i=1,\cdots,l, 0\leq p\leq 1$, *then*

$$l^{p-1}\left(\sum_{i=1}^{l}a_i^p\right)\leq\left(\sum_{i=1}^{l}a_i\right)^p.$$

PROPOSITION 2.3. *Let* $0<p<\infty$, $q\geq 0$, *and let* $f(z)=\sum a_\alpha z^\alpha$ *be in* $O(B)$. *Then for any positive integer* m *we have, with* $l=\max\{1,p\}$

$$(2.2)\quad M_1^p(f:\rho)\geq c_p\rho^{pm}(m+1)^{l(1-n)+pq/2}\sum_{|\alpha|=m}\left(\frac{\alpha!}{\Gamma(n+m+q)}\right)^{p/2}|a_\alpha|^p$$

where c_p *is independent of* f.

Proof. We use the polar coordinates to get

$$\int f(\rho z)\bar{z}^\alpha dv_q=\frac{2\Gamma(n+q)}{\Gamma(n)\Gamma(q)}\int_0^1 r^{2n-1}(1-r^2)^{q-1}dr\int f(\rho rz)(\bar{r}\bar{z})^\alpha d\sigma(z)$$
$$=a_\alpha\rho^{|\alpha|}\frac{\alpha!\Gamma(n+q)}{\Gamma(n+|\alpha|+q)}$$

For $|\alpha|=m$, we have

$$\rho^m \alpha! a_\alpha = 2\frac{\Gamma(n+m+q)}{\Gamma(n)\Gamma(q)} \int_0^1 r^{2n-1}(1-r^2)^{q-1} dr \int f(\rho r z) \bar{z}^\alpha d\sigma(z)$$

It follows that

$$\rho^m \alpha! |a_\alpha| = \frac{\Gamma(n+m+q)}{\Gamma(n)\Gamma(q)} \int_0^1 s^{n+m/2-1}(1-s)^{q-1} ds \times \int |f(\rho\sqrt{s}z)| \cdot |z^\alpha| d\sigma(z).$$

We multiply both sides by $(m!/\alpha!)^{1/2}$ and sum over all indices α with $|\alpha|=m$ to obtain

$$\rho^m \sum_{|\alpha|=m} (m!\alpha!)^{1/2} |a_\alpha| \leq \frac{\Gamma(n+m+q)}{\Gamma(n)\Gamma(q)} \int_0^1 s^{n+m/2-1}(1-s)^{q-1} ds \times \int |f(\rho\sqrt{s}z)| \left(\sum_{|\alpha|=m} (m!/\alpha!)^{1/2} |z^\alpha|\right) d\sigma(z).$$

By the Cauchy-Schwarz inequality and the fact that $M_1(f:r)$ *is* an increasing function of r, we have

$$(2.3) \quad \rho^m \sum_{|\alpha|=m} (m!\alpha!)^{1/2} |a_\alpha| \leq \frac{\Gamma(n+m+q)}{\Gamma(n)} \binom{n+m-1}{m}^{1/2} \frac{\Gamma(n+m/2)}{\Gamma(n+m/2+q)} M_1(f:\rho)$$

Taking p—th power, we obtain by Lemma 2. 2,

$$\rho^{pm} \binom{n+m-1}{m}^{p-1} \sum_{|\alpha|=m} (m!\alpha!)^{p/2} |a_\alpha|^p \leq \left(\frac{\Gamma(n+m+q)}{\Gamma(n)}\right)^p \binom{n+m-1}{m}^{p/2} \left(\frac{\Gamma(n+m/2)}{\Gamma(n+m/2+q)}\right)^p M_1^p(f:\rho),$$

which, by Sterling's formula, is equivalent to

$$M_1^p(f:\rho) \geq C\rho^{pm}(m+1)^{-n+1+pq/2} \sum_{|\alpha|=m} \left(\frac{\alpha!}{\Gamma(n+m+q)}\right)^{p/2} |a_\alpha|^p$$

for some C independent of f. This gives (2. 2) for $p\leq 1$, and the case $p>1$ is dealt with similarly.

Now we prove the main theorem. First assume $0<p<1$. Then by Theorem 1. 1,

$$C\|f\|_{p,q}^p \geq \int_0^1 (1-\rho)^{n+q-np-1} M_1^p(f:\rho)\,d\rho$$

$$\geq \sum_{m=0}^{\infty} \int_{1-1/m+1}^{1-1/m+2} (1-\rho)^{n+q-np-1} M_1^p(f:1-1/(m+1))\,d\rho$$

$$= \sum_{m=0}^{\infty} \frac{(m+1)^{np}(m+2)^{(n+q)} - (m+2)^{np}(m+1)^{n+q}}{(m+1)^{n+q}(m+2)^{n+q}} M_1^p(f:1-1/(m+1))$$

But $(m+1)^{np}(m+2)^{(n+q)} - (m+2)^{np}(m+1)^{n+q}$

$$\geq (m+2)^q[(m+1)^{np}(m+2)^n - (m+2)^{np}(m+1)^n]$$

$$= (m+2)^{q+n}(m+1)^{np}[1-t^{1-p}],$$

where $t=[(m+1)/(m+2)]^n$ and since $1-t^{1-p} \geq (1-p)(1-t)$, above sum is not less than

$$(1-p)\sum_{m=0}^{\infty} \frac{(m+1)^{np}[(m+2)^n - (m+1)^n]}{(m+1)^{n+q}(m+2)^n} M_1^p(f:1-1/(m+1))$$

$$\geq n(1-p)\sum_{m=0}^{\infty} \frac{(m+1)^{np-1-q}}{(m+2)^n} M_1^p(f:1-1/(m+1)).$$

Since $(1-1/(m+1))^{mp} \to e^{-p}$ as $m \to \infty$, Proposition 2.3 gives the result for $0<p<1$. For $p=2$, the inequality becomes Parseval's identity. We shall use Marcinkiewcz interpolation theorem to obtain the result for $0<p\leq 2$.

Define a space of sequences $\{l_q^p(Z_+^n), d\nu\}$ with $\nu(\alpha)=(|\alpha|+1)^{-2(n+q/2)}$ for $\alpha \in Z_+{}^n$, and define $(Tf)(\alpha)=(|\alpha|+1)^{n+q/2}(\alpha!/\Gamma(n+|\alpha|+q))^{1/2}|a_\alpha|$ for $f\in A_q^p$. Then for $0<p<1$,

$$\nu\{\alpha : |(Tf)(\alpha)|>s\} = \sum_{(Tf)(\alpha)>s} (|\alpha|+1)^{-2(n+q/2)}$$

$$\leq 1/s^p \sum_{\alpha\geq 0} (|\alpha|+1)^{(n+q/2)(p-2)} (\alpha!/\Gamma(n+|\alpha|+q))^{p/2} |a_\alpha|^p.$$

Therefore, T as a mapping of $\{A_q^p, dv_q\}$ into $\{l_q^p(Z_+{}^n), d\nu\}$, is of weak type (p,p) for $0<p<1$. On the other hand, T is of strong type (2.2). It follows that T is of strong type (p,p) for $0<p\leq 2$, which is (2.1).

The following dual result is easily obtained by considering the orthogonal projection.

COROLLARY 2.4. *Let* $2\leq p<\infty$, $q\geq 0$ *and let* $f(z)=\sum a_\alpha z^\alpha$ *be in* $O(B)$. *Then*

$$\|f\|_{p,q}^p \leq c_p \sum_{\alpha \geq 0} (|\alpha|+1)^{(n+q/2)(p-2)} \left(\frac{\alpha!}{\Gamma(n+|\alpha|+q)} \right)^{p/2} |a_\alpha|^p$$

for some c_p independent of f.

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