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THE STRUCTURE OF REGULARITY OF NEAR-RINGS

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1. Introduction

The concepts of regularity of near-rings have been studied by many authers, J.C. Beidleman, S. Ligh, D.Z. Chao, H.E. Heatherly and A. Oswald, their main results are in G. Pilz [12].

In 1980, G. Mason [9] introduced the notions of strong regularity of near-ring. He proved that for any zero-symmetric near-ring with identity, the concepts of left regularity, strong left regularity and strong right regularity of near-rings are equivalent, and in 1984, the notion of strong regularity of near-rings has been studied further by C.V.L.N. Murty [10].

The Von Neumann regularity of rings and generalizations were studied by J.W. Fisher, R.L. Snider [4], Y. Hirano, H. Tominaga [5] and T.R. Savage [14].

In 1985, M. Ôhori [11] studied the characterization of strong π-regularity of rings and also in 1985 L. Li and B.M. Schein [7].

In section 3, we will generalize or improve some of the main results of Beidleman and Ligh in [12] and we will introduce the concepts π-regularity of near-rings which are more general concepts of regularity of near-rings.

Characterization of strong regularity and strong π-regularity of near-rings are investigated, and the purpose of this section is to prove that the concepts of left regularity, strong left regularity, strong right regularity, π-regularity and strong π-regularity of near-rings with some conditions are equivalent for arbitrary near-rings.

In section 4, semi π-regularity and κ-regularity will be introduced, and we shall characterize relationships between them. Finally, we will investigate their equivalent condition under zero-symmetric reduced near-ring.

In this paper, N will represent a (right) near-ring. A near-ring N is said to be zero-symmetric if $a0=0$ for all a in N and reduced if it has no non-zero nilpotent element.

We refer the reader to the book by G. Pilz [12] and [13] for the remainder definition and basic concepts of near-ring theory.

2. Regularity and π-regularity of near-rings

THEOREM 3. 1. *Let N be a near-ring. N is regular if and only if for each $a\in N$, there exists an idempotent $e\in N$ such that $a\in Na$ and $Na=Ne$.*

Proof. Suppose N is regular and let $a\in N$. Then there exists $x\in N$ such that $a=axa$. Since xa and ax are idempotents in N, take $xa=e$, we have $a=axa\in Na$ and $Na=Naxa\subset Nxa=Ne$. Hence $Na=Ne$.

Conversely, assume the conditions. Let $a\in N$, by assumption, there exists an idempotent e in N such that $a\in Na$ and $Na=Ne$.

Now, since $a\in Ne$, $a=xe$ for some element x in N, and $e=ee\in Ne=Na$, so $e=ya$ for some element y in N. It follows that $a=xe=xee=xeya=aya$. Hence N is reqular.

COROLLARY 2. 2. (Beidleman [1], Ligh [8], [12]) *Let N be a near-ring with identity. N is regular if and only if for each a in N there exists an idempotent e in N such that $Na=Ne$.*

THEOREM 2. 3. *Every regular near-ring contains no non-zero nil N-subgroups.*

COROLLARY 2. 4. (Beidleman [1], [12]). *Any regular near-ring with identity contains no non-zero nil N-subgroups.*

A near-ring N is said to be π-regular if for every a in N, there exists x in N and exists positive integer n such that $a^n=a^nxa^n$. Such an element a is called π-regular.

A near-ring N is called left (right resp.) π-regular, if for each element $a\in N$, there exists a positive integer n, such that a^n is left (right resp.) regular.

REMARK 2. 5. Every regular near-ring is π-regular. But not conversely.

THEOREM 2. 6. *Let N be a near-ring. N is π-regular if and only if* *for any a in N, there exists a positive integer n, and idempotent e in N such that* $a^n \in Na^n$ *and* $Na^n = Ne$.

REMARK 2.7. A π-reqular near-ring may have non-zero nil N-subgroups.

The set $Z(N) = \{x \in N | ax = xa$ for each $a \in N\}$ is called the center of a near-ring N. If N is distributive near-ring, then $Z(N)$ is a subnear-ring of N.

THEOREM 2.8. *The center of a distribut ve* π*-regular near-ring is also* π*-regular.*

Proof. Let N be a π-regular near-ring and let $a \in Z(N)$. Then there exists x in N, exists a positive integer n such that $a^n = a^n x a^n$, so $a^n = a^n x a^n x a^n$. We will show that $x a^n x \in Z(N)$.

Let $c \in N$, since $a^n \in Z(N)$, $a^n x \in Z(N)$. Indeed, $(a^n x) c = a^n (xc) = (xc) a^n = (xc) a^n x a^n = a^n (xcx) a^n = a^n x a^n c x = a^n c x = c (a^n x)$.

Similarly, $x a^n \in Z(N)$. Now, $(x a^n x) c = x (a^n x) c = xc (a^n x) = x (a^n c) x = (a^n x) c x = c (a^n x) x = c (x a^n) x = c (x a^n x)$. Hence $Z(N)$ is π-regular.

COROLLARY 2.9. *The center of a distributive regular near-ring is also regular.*

REMARK 2.10. $(A_4, +)$ is alternating group on 4-letters which is not abelian. We can define multiplication $\cdot$ such that $(A_4, +, \cdot)$ is a distributive π-regular near-ring, but not a ring.

3. Strong regularity and Strong π-regularity of near-rings

N is said to be strongly π-regular if it is both left and right π-reqular, and strongly left (right resp.) π-regular if for any element $a \in N$ there exists a positive integer n, such that a^n is strongly left (right resp.) regular.

LEMMA 3.1. *Let N be a left (or right) regular near-ring. If for any a, b in N with* $ab = 0$ *then* $(ba)^n = b0$, *for all positive integer n.*

LEMMA 3.2. *Let N be a left (or right) regular near-ring. If for any a, b in N with* $ab = 0$ *and* $a^n = a0$, *for all positive integer* $n \geq 2$, *then* $a = 0$. *In this case, if N is zero-symmetric then N is reduced.*

LEMMA 3.3. (G. Mason [9]). *Let N be a zero-symmetric, reduced near-ring. If for any a, b in N with* $ab = 0$, *then* $ba = 0$ *and N has I.F.P.*

LEMMA 3.4. *Let N be a zero-symmetric near-ring with left identity. If N is reduced, then every idempotent is central.*

LEMMA 3.5. *If N is a left regular near-ring, then N is regular. Moreover, if $a=xa^2$ for some x, a in N, then $ax=xa$.*

Proof. Let $a\in N$. Since N is left regular, $a=xa^2$ for some x in N. $(a-axa)a=0$, by Lemma 3.1, $a(a-axa)=a0$.

Hence $(a-axa)^2=a0-axa0=(a-axa)0$. So, by Lemma 3.2, $a=axa$. Therefore N is regular.

Next, since $(ax-xa)a=0$, $a(ax-xa)=a0$. Thus we have $(ax-xa)^2 =ax0-xa0=(ax-xa)0$. Consequently $ax=xa$.

COROLLARY 3.6. *If N is left regular then N is right regular, furthermore, it is strongly right regular.*

LEMMA 3.7. *If N is either a left or right regular near-ring, then, for any $a\in N$ and any $e^2=e\in N$ such that $ea^n=ea^ne$ for all positive integer n.*

THEOREM 3.8. *If N is left π-regular and right regular, then N is π-regular and if $a^n=xa^{2n}$ for some a, x in N and some positive integer n, then $a^nx=xa^n$.*

Proof. Let $a\in N$. Since N is left π-regular, there is an element $x\in N$, such that $a^n=xa^{2n}$ for some positive integer n.

Form $(a^n-a^nxa^n)a^n=0$, $a^n(a^n-a^nxa^n)=a^n0$ by Lemma 3.1.

So, $(a^n-a^nxa)^2=a^n(a^n-a^nxa^n)-a^nxa^n(a^n-a^nxa^n)=(a^n-a^nxa^n)0$.

By Lemma 3.2, $a^n=a^nxa^n$. Hence N is π-regular.

Next, since $a^n=a^nxa^n=xa^{2n}$, $(a^nx-xa^n)a^n=0$ and $(a^nx-xa^n)a^nx=0$.

Then $a^n(a^nx-xa^n)=a^n0$ and $a^nx(a^nx-xa^n)=a^nx0$. Hence $(a^nx-xa^n)^2 =a^nx0-xa^n0=(a^nx-xa^n)0$. Therefore $a^nx^n=xa^n$.

COROLLARY 3.9. *If N is left π-regular and right regular, then N is strongly left π-regular and strongly right π-regular.*

THEOREM 3.10. *Let N be any arbitrary near-ring. The following statements are equivalent.*

(1) *N is left regular.*

(2) *N is strongly regular.*

(3) *N is right regular and left π-regular.*

(4) *N is right regular and strongly left π-regular.*
(5) *N is right regular and strongly π-regular.*
(6) *N is strongly left regular.*
(7) *N is strongly right regular.*
(8) *N is regular and for any idempotent $e\in N$ and any $a\in N$ such that $ea^n=ea^ne$ for all positive integer n.*
(9) *N is regular and for any idempotent $e\in N$ and any $a\in N$ such that $ea=eae$.*

Proof. (1)$\Leftrightarrow$(2). By Lemma 3.5. (2)$\Rightarrow$(3). Since left regular is left π-regular. (3)$\Rightarrow$(4). By Corollary 3.9. (4)$\Rightarrow$(5). By Theorem 3.8.

(5)$\Rightarrow$(1). Suppose N is right regular and strongly π-regular.

Let $a\in N$. Since N is left π-regular, there exists an element $x\in N$ such that $a^n=xa^{n+1}$ for some positive integer n.

If $n=1$ or $n=2$, clearly N is left regular. It suffice to show the case positive integer $n>2$. $(a^{n-1}-xa^n)a=0$ and $(a^{n-1}-xa^n)a^{n-1}=0$.

By Lemma 3.1, $a(a^{n-1}-xa^n)=a0$ and $a^{n-1}(a^{n-1}-xa^n)=a^{n-1}0$.

Now $(a^{n-1}-xa^n)^2=a^{n-1}(a^{n-1}-xa^n)-xa^n(a^{n-1}-xa^n)=a^{n-1}0-xa^n0=(a^n-xa^n)0$. By Lemma 3.2, $a^{n-1}=xa^n$.

Continuing this procedure, we obtain that $a=xa^2$. Therefore N is left regular. (1)$\Rightarrow$(6)$\Rightarrow$(7)$\Rightarrow$(8)$\Rightarrow$(9).

These implications follow from the previous Lemma 3.5 and Lemma 3.7.

Finally, we will prove that (9)$\Rightarrow$(1).

Suppose N is regular and for any $a'\in N$, and idempotent $e\in N$ such that $ea'=ea'e$.

Let $a\in N$. Since N is regular, $a=axa$ for some $x\in N$.

Then $a=axa=axaxa=a(xax)a=a(xaxxa)a=axaxxa^2=ax^2a^2=ya^2$, where $y=ax^2$. Hence N is left regular.

4. Semi π-regularity and κ-regularity of near-rings

DEFINITION 4.1. A near-ring N is said to be left semi π-regular if for each $a\in N$, there exists $x\in N$ such that $a^n=axa^n$ for some positive integer n.

Analogously for right semi π-regular.

Such an element a is called left (right, resp.) semi π-regular.

A near-ring N is called left (right, resp.) κ-regular if for every $a\in N$, there exists $x \in N$ such that $a^n = xa^{n+1}$ ($a^n = a^{n+1}x$, resp.) for some positive integer n.

REMARK 4.2. (1) Every π-regular near-ring is left (and, right) semi π-regular.

(2) Every left (right, resp.) π-regular near-ring is left (right, resp.) semi π-regular.

(3) Every right (left, resp.) π-regular near-ring is also right (left, resp.) κ-regular.

(4) Every right (left, resp.) κ-regular near-ring is also right (left, resp.) semi π-regular.

There exist many examples of semi π-regularity and κ-regularity of near-rings, we can easily see for finite near-rings.

THEOREM 4.3. *If N is a left and right κ-regular near-ring, then N is π-regular.*

Proof. Let $a \in N$. Since N is left and right κ-regular, there exists x and y in N such that $a^n = xa^{n+1}$, $a^m = a^{m+1}y$ for some positive integers n and m. Thus, $a^n = xa^{n+1} = x(xa^{n+1})a = x^2a^na^2 = x^2(xa^{n+1})a^2 = x^3a^na^3 \cdots = x^ma^na^m = x^ma^{n+m}$.

Analogously, from $a^m = a^{m+1}y$, $a^m = aa^my = a(a^{m+1}y)y = a^{m+2}y^2 = \cdots = a^{m+n}y^n$.

Consequently, $a^{n+m} = a^{m+n} = a^ma^n = a^{m+n}y^nx^ma^{n+m} = a^{n+m}za^{n+m}$, where $z = y^nx^m$. Hence N is π-regular.

THEOREM 4.4. *If N is a left semi π-regular and right regular near-ring, then N is regular.*

THEOREM 4.5. *If N is π-regular and right regular, then N is right κ-regular.*

Proof. Let $a \in N$. Since N is π-regular, there is an element $x \in N$ such that $a^n = a^nxa^n$ for some positive integer n. Form $(a^{n-1} - a^nxa^{n-1})a = 0$, $a(a^{n-1} - a^nxa^{n-1}) = a0$ and $(a^{n-1} - a^nxa^{n-1})^2 = (a^{n-1} - a^nxa^{n-1})0$. Thus we have that $a^{n-1} = a^nxa^{n-1}$.

Continuing this process, it follows $a = a^nxa$.

Hence $a^m = a^{m-1}a = a^{m-1}a^nxa = a^{m+1}(a^{n-2}xa) = a^{m+1}y$, where $y = a^{n-2}xa$.

Therefore N is right κ-regular.

THEOREM 4.6. *Let N is zero-symmetric with left identity, and reduced.*

The following statements are equivalent.

(1) *N is π-regular.*

(2) *N is left κ-regular.*

(3) *N is left semi π-regular.*

(4) *N is regular.*

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