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ON MULTIPLICATION NON-COMMUTATIVE SEMIGROUPS

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0. Introduction

Multiplication rings have been extensively studied by Gilmer and Mott [1], Griffin[2], P.F. Smith[5], etc., and have been characterized using powerful ideal-theoretic techniques. Their techniques are not enough to characterize multiplication semigroups, since multiplicatication semigroups are not necessarily of dimension less than or edual to one, unlike multiplication ring (the term "dimension" is defined exactly the same way as in ring).

V.L. Mannepall[3], [4] investigated the structure of multiplication commutative semigroups containing identity. The aim of this paper is to characterize the structures of multiplication non-commutative semigroups under the some conditions.

1. Characterization of P-primary ideals in a noncommutative semigroups

Troughout this paper, S will denote a semigroup. For subsets A and B of S and $x \in S$, we adopt the following notations:

$AB = \{ab \mid a \in A,\ b \in B\}$,
$A \leq B$: A is a subset of B,
$A < B$: A is a proper subset of B
(x) : the principal ideal generated by x in S.

For any ideal A in S,

$\bar{A} = \{x \in S \mid (x)^n \leq A$ for some positive integer $n\}$
$r(A)$ = The intersection of all prime ideals containing A
$A^* = \{x \in S \mid x^n \in A$ for some positive integer $n\}$

In any semigroup, $\bar{A} \subseteq r(A) \subseteq A^*$, [6].

DEFINITION. A semigroup S is a left (right) multiplication semigroup if for all ideals $A<B$ of S, there exists an ideal C such that $A=BC$ (resp. $A=CB$). A semigroup S is a multiplication semigroup if S is both a left and a right multiplication semigroup.

DEFINITION. Let P be a prime ideal of S. An ideal A in S is said to be left (right) P-primary if

(i) Let X, Y be ideals in S such that $XY\leq A$ and $Y\nleq A (X\nleq A)$ then $X\nleq r(A)$ $(Y\nleq r(A))$ and

(ii) $r(A)=P$.

It is easily showed that a primary ideal need not be a power of a prime ideal and conversely, a power of a prime ideal need not be primary.

(1L) Given ideals $A<P$ of a semigroup S with P prime, there exists an ideal B such that $A=PB$.

LEMMA 1.1. *Let S be a semigroup which satisfies* (1L) *and $P<I$ ideals of S with P prime. Then $P=IP$.*

Proof. The proof follows as in Theorem 1.1 of [5].

Let I be an ideal of a semigroup S, we denote $I^{w}=\bigcap_{n=1}^{\infty} I^{n}$.

LEMMA 1.2. *Let S be a semigroup which satisfies* (1L), *and $A<P$ ideals of S with P prime. Then $A\leq P^{w}$ or $A=P^{k}$ or $A=P^{k}B$ for some positive integer k and for some ideal $B\nleq P$.*

Proof. Suppose $A\nleq P^{w}$. There exists $n\in N$ such that $A\leq P^{n}$ but $A\nleq P^{n+1}$. The result is proved by induction on n.

COROLLARY 1.3. *Let S be a semigroup which satisfies* (1L), *and A, P ideals of S with P prime and $A\nleq P^{w}$. Then there exists $k\geq 0$ such that $A\leq P^{k}$ and $P^{k+1}=AP\cup P^{k+2}$.*

Proof. By Lemma 1.2, if $A<P$ then $A=P^{n}$ or $A=P^{n}B$ for some positive integer n and for some ideal $B\nleq P$. If $A=P^{n}$, then $P^{n+1}=P^{n}P=AP$, Thus $P^{n+1}=AP\cup P^{n+2}$. If $A=P^{n}B$ and $B\nleq P$, then $P<B\cup P$. By Lemma 1.1, $P=(B\cup P)P=BP\cup P^{2}$. Hence $P^{n+1}=P^{n}P=P^{n}(BP\cup P^{2})=AP\cup P^{n+2}$. If $A\nleq P$, then $P<A\cup P$. By Lemma 1.1, $P=(A\cup P)P=AB\cup P^{2}$.

LEMMA 1.4. *Let S be a semigroup which satisfies (1L) and P a prime ideal of S. Then P^w is a prime ideal of S or $P^n=P^{n+1}$ for some positive integer n.*

COROLLARY 1.5. *Let S be a semigroup which satisfies (1L) and P a minimal prime ideal of S. Then $P^n=P^{n+1}$ for* some n.

COROLLARY 1.6. *Let S be a semigroup which satisfies (1L) and P a prime ideal of S. Then $P^w=PP^w$.*

LEMMA 1.7. *Let S be a semigroup which satisfies (1L), P a prime ideal and n a positive integer. Then any ideal A with $P^n \leq A \leq P$ is left P-primary.*

Proof. Since $A \leq P$, $r(A)=\bigcap P_i \leq P$. Since $P^n \leq A$, for any $p \in P$, $(p)^n \leq P^n \leq A$. Thus $P \leq r(A) \leq P$ implies $P=r(A)$. Let X, Y be ideals of S such that $XY \leq A$ and $Y \not\leq A$. Suppose $X \not\leq r(A)=P$, then $P<X \cup P$. By Lemma 1.1, $P=(X \cup P)P=XP \cup P^2<X \cup P$ and $P=(X \cup P)(XP \cup P^2) \leq XP \cup P^3 \leq P$. Hence $P=XP \cup P^2=XP \cup P^3=\cdots=XP \cup P^n$ implies $P=XP \cup A$. Since $XY \leq A \leq P$, $Y \leq P$. Hence there exists B such that $Y=PB$. Thus $Y=PB=(XP \cup A)B=XY \cup AB \leq A$, a contradiction.

(2L) Given any positive integer n, and ideals $A<P^n$ of S with P prime, there exists an ideal B of S such that $A=P^nB$.

It is clear that (2L) implies (1L).

PROPOSITION 1.8. *Let S be a semigroup such that (1L). Suppose $P=PA$ for all ideals $P<A$ with P prime. Then S satisfies (2L).*

Proof. Let A, P be ideals of S and n a positive integer such that $A<P^n$. When $n=1$, $A=PB$ for some ideal B. Suppose for $n-1$, $A=P^{n-1}C$ for some ideal C. If $C<P$ or $C=P$, then it is clear. If $C \not\leq P$, then $P<C \cup P$. Using Lemma 1.1, $P=PC \cup P^2$. Suppose $P^{k-1}=P^{k-1}C \cup P^k$, then $P^k=P^kC \cup P^{k+1}$. Hence $P^{n-1}=A \cup P^n \subseteq P^n$. Thus $P^{n-1}=P^n$ and $A=P^nC$.

Lemma 1.1 and Proposition 1.8 show that the conditions (1L) and (2L) are equivalent for a commutative semigroup S.

LEMMA 1.9. *Let P be a prime ideal of a semigroup S such that (1L). If $B \leq P<A$ with P prime, then $B=AB$ i.e., $B \leq A^w$.*

Proof. By Lemma 1.1, $P=AP$. Since $B<P$, $B=PD$ for some ideal D of S. Hence $B=(AP)D=A(PD)=AB$. Moreover $B=AB=A(AB)=\cdots=A^{w}B\leq A^{w}$.

THEOREM 1.10. *If a semigroup S satisfies condition $(1L)$. Then S satisfies that each ideal whose radical is prime is primary. Morever P^n is P-primary for each n.*

Proof. Suppose Q is an ideal of S such that $r(Q)=P$ is prime. If $P=S$, then Q is primary. If $P<S$. Suppose Q is not primary. Then there exists $p\in P-Q$ and $s\in S-P$ such that $sp\in Q$. Let $A=Q\cup(p)P$. Then $p\notin A$. For, suppose $p\in A$, then $p\in A-Q$. Since there exists $b\in P$ such that $p=pb$, $p=pb=pb^2=\cdots=pb^n\in Q$, a contradiction. By Lemma 1.9, $A\cup(p)=(A\cup(p))(P\cup(s))$. Hence $A\cup(p)\leq AP\cup(p)P\cup(s)A\cup(p)(s)\leq A$, a contradiction.

THEOREM 1.11. *Let S be a semigroup with identity $(1L)$ and $P=PA$ for any ideals $p<A$ with P prime. Then every left (right) primary ideal is a power of its radical.*

Proof. Let Q be a left primary ideal with $r(Q)=P$. If $P=S$, then $Q=S$. Assume $P\neq S$. If $Q\leq P^w$ and $P^n\neq P^{n+1}$ for any n, By Lemma 1.4, P^w is prime. It follows that $P\leq P^w$, a contradiction. If $Q\leq P^w$ and $P^n=P^{n+1}$, for some positive integer n. For any $x\in P^n$, $(x)\leq P^n$. By Proposition 1.8, $(x)=P^nB$ for some ideal B. Thus $(x)=P(x)$ implies there exists $t\in P$ such that $x=tx$. Hence $x=tx=\cdots=t^kx\in Q$ implies $Q=P^n$. Now, when $Q\not\leq P^w$, by Lemma 4.4, $Q=P^k$ or $Q=P^kB$ and $B\not\leq P$. Since Q is P-primary, $P^k\leq Q$. Hence $Q=P^k$.

2. Multiplication non-commutative semigroups

Let S be a semigroup which satisfies (1L) and let $\rho_I=(I\times I)\cup I_S$ be a Rees congrunce. Then $S/\rho_I=\{I\}\cup\{\{x\}\mid x\in S\backslash I\}$ is a Rees quotient semigroup. Let P be a minimal prime ideal of A in S. Then P/ρ_A is a minimal prime ideal of S/ρ_A, since $\{I\mid I$ is an ideal of S containing $A\}$ is one-to-one corresponding to $\{I/\rho_A\mid I/\rho_A$ is an ideal of $S/\rho_A\}$. Since S satisfies (1L), S/ρ_A also satisfies (1L). By Corollary 1.5, there exists n such that $(P/\rho_A)^n=(P/\rho_A)^{n+1}$. It follows that $A\cup P^n=A\cup P^{n+1}$. Let $\eta_A(P)$ denote the least positive integer n such that $(P/\rho_A)^n=(P/\rho_A)^{n+1}$

and $P(A)$ be a collection of a minimal prime ideal of A. Define $\hat{A}=\bigcap_{P\in P(A)}(A\cup P^{\eta_A(P)})$.

(TL) Every ideal A of a semigroup S is an intersection of left primary ideals.

Every ideal in a (left, right) duo Neotherian semigroup has a reduced (left, right) primary decomposition [1].

THEOREM 2. 1. *Let S be a semigroup with identiy which satisfies (TL) and $(1L)$, antd A an ideal of S. Then $\hat{A}=A$.*

Proof. By property (TL), there exist families of prime ideals P_λ and ideals A_λ such that A_λ is a left P_λ-primary for each λ in A and $A=\bigcap_{\lambda\in A}A_\lambda$. By Theorem 1. 11, there exists a positive integer k such that $P_\lambda^k=A_\lambda$. Let $\lambda\in\Lambda$, then there exists $P\in P(A)$ such that $P\leq P_\lambda$. Hence $(P/\rho_A)^w\leq(P_\lambda/\rho_A)^w$. Since $A_\lambda=P_\lambda^k$ and $(P/\rho_A)^n=(P/\rho_A)^{n+1}$ for some positive integers k, n, $(P/\rho_A)^n\leq(P_\lambda/\rho_A)^w\leq P_\lambda^k/\rho_A=A_\lambda/\rho_A$. Hence for each λ, $A\cup P^n\leq A_\lambda$. It follows that $\bigcap_{P\in P(A)}(A\cup P^{\eta_A(P)})\leq\bigcap A_\lambda=A$ implies $\hat{A}=A$.

THEOREM 2. 2. *Let S be a semigroup with identity which satisfies $(2L)$. Then $\hat{A}=\bigcap_{P\in P(A)}P^{\eta_A(P)}$ for all ideal A of S.*

Proof. $P^{\eta_A(P)}\leq A\cup P^{\eta_A(P)}\leq P$. By Lemma 1. 7, $A\cup P^{\eta_A(P)}$ is P-left primary. By Theorem 1. 11, $A\cup P^{\eta_A(P)}=P^k$ for $1\leq k\leq\eta_A(P)$. Hence $A\cup P^{\eta_A(P)}=A\cup P^k$ implies $(P/\rho_A)^{\eta_A(P)}=(P/\rho_A)^k$.

DEFINITION. Let S be a semigroup and A, B ideals of S. Then A and B are called incomparable if $A\nleq B$ and $B\nleq A$.

LEMMA 2. 3. *Let S be a semigroup with $(1L)$ and let P and Q be incomparable prime ideals of S. Then $PQ=QP=P\cap Q$.*

LEMMA 2. 4. *Let S be a semigroup with $(2L)$ and (TL). Let P be a prime ideal of S and n a positive integer. Then no ideal A satisfies $p^{n+1}<A<P^n$.*

PROPOSITION 2. 5. *Let S be a semigroup with identity which satisfies (TL) and $(2L)$. Let A be an ideal of S and P a minimal prime ideal of A such that $n=\eta_A(P)$. Then there exists an ideal B such that $A=$*

$P^nB=BP^n=P^n\cap B$ or $A=P^n$.

Proof. By Theorem 1.11, $\hat{A}=A=\bigcap_{P\in P(A)} P^{\eta_A(P)}$, it follows that $A\leq P^n$. Suppose $A<P^n$. By Proposition 1.8, there exists an ideal C such that $A=P^nC$. Let $B=\{s\in S\mid P^ns\leq A\}$. Then $C\leq B$. Hence $A=P^nC\leq P^nB\leq A$ implies $A=P^nB$. Since $A=P^n\cap D$ where $D=\bigcap_{T\in P(A)-\{P\}} T^{\eta_A(T)}$, $P^nD\leq A$ implies $D\leq B$. For each T in $P(A)-\{P\}$, $A=P^nB\leq T^{\eta_A(T)}$ and $T^{\eta_A(T)}$ is T-primary, by Theorem 1.10. Since $P^n\not\leq T$, by Lemma 2.4, $B\leq T^{\eta_A(T)}$. Hence $B\leq D$ implies $B=D$ and $A=P^n\cap B$. Let $A'=BP^n\leq B\cap P^n=P^nB=A$. Then there does not exists $A'\leq Q<A\leq P^n<P$. Thus P is a minimal prime ideal of A and $\eta_{A'}(P)=n$. By Theorem 2.2, $A'=P^n\cap\{\bigcap_{H\in P(A')-\{P\}} H^{\eta_{A'}(H)}\}$. For each $H\in P(A')-\{P\}$, $A\leq H^{\eta_{A'}(H)}$ by Lemma 1.1, Lemma 2.2 and Lemma 2.4. Consequently $A\leq A'$ and $A=A'$.

LEMMA 2.6. *Let S be a semigroup such that for all ideals $A<B$ with A principal, there exists an ideal C such that $A=BC$. Then S is a left multiplication semigroup.*

THEOREM 2.7. *Let S be a semigroup with identity which satisfies (TL) and $(2L)$. Then S is a left multiplication semigroup.*

Proof. Using Lemmas 1.1, 2.4, 2.5, Proposition 2.5, Theorems 1.11, 2.1, 2.2, the proof follows as in Theorem 2.9 of [7].

(TR) Every ideal A of a semigroup S is an intersection of left primary ideals.

(2R) Given any positive integer n, and ideals $A<P^n$ of S with P prime, there exists an ideal B of S such that $A=BP^n$.

THEOREM 2.8. *Let S be a semigroup with identity with (TR) and $(2R)$. Then S is right multiplication semigroup.*

THEOREM 2.9. *Let S be a semigroup with identity such that every ideal in S is an intersection of primary ideals. Then S is an multiplication semigroup iff S satisfies $(1L)$ and left primary ideals are right primary.*

Proof. Suppose S is an multiplication semigroup. Then S satisfies (1L). By Lemma 1.1, for ideals $P<A$ with P prime, $P=PA$. By Theorem 1.11, and Lemma 1.7, every left primary ideals is right primary.

Conversely, let $P<A$ be ideals of S with P prime. Then $P^2 \leq PA \leq P$. By Lemma 1.7, PA is left P-primary. By hypotheses, PA is right P-primary. Since $A \nleq P$, $P \leq PA$. Hence $P=PA$. Thus S is an multiplication semigroup.

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