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THE BEHAVIOR OF THETA-SERIES UNDER SLASH OPERATORS*

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0. Introduction

It is well known that the representations between integral quadratic forms are closely related to Siegel's modular forms. But unfortunately, our knowledge on half integral weight Siegel's modular forms is substantially limited compared to that on integral ones and thereby we have not been so succesful in applying modular form theory to the representations of integral quadratic forms in an odd number of variables. This is simply because the former is far more difficult to handle than the latter. Recently though, a great deal of work has been done in this direction since Shimura [6] made the crucial contribution by obtaining Euler product for the classical modular forms of half integral weight in 1973.

In 1975, Andrianov-Maloletkin [2] found a transformation formula for the theta-series of any degree associated to a given positive definite integral quadratic form in an even number of variables. Lately Carlsson-Johannßen [3] and Styer [8] independently succeeded in obtaining transformation formulae when a given quadratic form has an odd number of variables. Although their transformation formulae are very important results by themselves, the formulae are rather complicated and not so practical because the formula of Carlsson-Johannßen requires a reduction process due to Andrianov-Maloletkin and that of Styer's requires so called the Dirichlet's theorem for coprime symmetric pairs of matrices.

In this article, the result of Andrianov-Maloletkin is extended to half integral weight theta-series in a different manner which turns out to be very simple and practical in many applications. In fact, the transformation formula of Andrianov-Maloletkin is equivalent to the following: the theta-series of degree n of a given positive definite integral

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quadratic form in m variables, m even, is contained in the space $M_k^n(q, \chi)$ of Siegel's modular forms of degree n, level q, weight $k=m/2\in Z$, with a Dirichlet character χ modulo q, where q and χ are determined by the given quadratic form (Theorem 3.1). The purpose of this article is to generalize the equivalent result of Andrianov-Maloletkin under a slightly modified definition of $M_k^n(q,\chi)$ when $k=m/2$ is a half integer (Theorem 3.3). The most important advantage of this is that we can apply those Hecke operators developed by Zhuravlev [9, 10] to higher degree theta-series of half inegral weight.

Let $\boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}$, and $\boldsymbol{C}$ be the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively. $\boldsymbol{T}$ is the set of complex numbers of absolute value 1.

Let $M_{m,n}(A)$ be the set of all $m\times n$ matrices with entries from A, a commutative ring with 1, and let $M_n(A)=M_{n,n}(A)$. Let E_n and 0_n be the identity and zero matrices in $M_n(A)$. For $M\in M_m(A)$ and $N\in M_{m,n}(A)$, $M[N]$ means ${}^tNMN$ where ${}^tN$ is the transpose of N, For a $2n\times 2n$ matrix M, let A_M, B_M, C_M, and D_M be the $n\times n$ block matrices of M in the upper left, upper right, lower left, and lower right corners, repectively, and write $M=(A_M, B_M;\ C_M, D_M)$.

Let $G_n=GSp_n^+(\boldsymbol{R})=\{M\in M_{2n}(\boldsymbol{R}) : J_n[M]=rJ_n,\ \ r>0\}$, where $J_n=(0_n, E_n;\ -E_n, 0_n)$, $\Gamma^n=Sp_n(\boldsymbol{Z})=\{M\in M_{2n}(\boldsymbol{Z}) : J_n[M]=J_n\}$.

Finally, for $M\in M_n(\boldsymbol{C})$, $\det(M)$ denotes the determinant of M and $e(M)$ is $\exp(2\pi i\sigma(M))$, where $\sigma(M)$ is the trace of M and $i=\sqrt{-1}$.

1. The universal covering group

Let n be a positive integer≥ 1. We set $H_n=\{Z=X+iY\in M_n(\boldsymbol{C}) : {}^tZ=Z,\ \ Y>0\}$ where $Y>0$ means that Y is positive definite. It is called the Siegel's upper half plane of degree n. Let $G=G_n$. We define the action of G on H_n by

$$(1.1)\quad M\langle Z\rangle=(A_MZ+B_M)\cdot(C_MZ+D_M)^{-1} \text{ for } M\in G,\ \ Z\in H_n.$$

Let $\widetilde{G}=\{M, \alpha(Z)) : M\in G$ and $Z\in H_n\}$ where $\alpha(Z)$ is of the form $\alpha(Z)^2=t\cdot\det M^{-1/2}\cdot\det(C_MZ+D_M)$ for some $t\in\boldsymbol{T}$. $\widetilde{G}$ is a multiplicative group under the multiplication $(M, \alpha(Z))\cdot(N, \beta(Z))=(MN,\ \alpha(N\langle Z\rangle)\cdot\beta(Z))$ and is called the universal covering group of G. Let π be the projection of $\widetilde{G}$ onto the first component G. We define the action of $\widetilde{G}$ on H_n by $\zeta\langle Z\rangle=\pi(\zeta)\langle Z\rangle$ for $\zeta\in\widetilde{G}$, $Z\in H_n$ (see (1.1)).

REMARK 1.1. The universal covering group was first introduced by Shimura [6] for $SL_2(\boldsymbol{R})$ and later generalized to $GSp_n^+(\boldsymbol{R})$ by Zhuravlev [9].

Let q be a positive integer and let $\Gamma_0(q)=\Gamma_0^n(q)=\{M\in\Gamma^n : C_M\equiv 0 \pmod{q}\}$. Let $\theta^n(Z)=\sum e({}^tXXZ)$, where the summation is over $X\in M_{1,n}(\boldsymbol{Z})$, be the standard theta-function of degree n.

We now fix a branch of $\det(CZ+D)^{1/2}$ in such a manner that $\det(CZ+D)^{1/2}\to\varepsilon_{\mathrm{sign}(\det D)}\cdot|\det D|^{1/2}$ as $Z\to 0_n$ in H_n along with the imaginary axis, where $\mathrm{sign}(\det D)$ is 1 or -1, respectively, for $\det D>0$ or <0, and $\varepsilon_d=1$ or i, respectively, for $d\equiv 1$ or $-1 \pmod 4$. It is well known that if q is divisible by 4, then we have (see [5, p. 29-31] for instance)

$$(1.2)\quad \theta^n(M\langle Z\rangle)=\chi_\theta(M)\cdot\det(C_MZ+D_M)^{1/2}\cdot\theta^n(Z),\quad Z\in H_n,$$

where $\chi_\theta(M)$ is a fourth root of 1 for $M\in\Gamma_0(q)$.

REMARK 1.2. If $n=1$ and $M=(a,b;\ c,d)\in\Gamma_0^1(q)$, with $4|q$, then $X_\theta(M)=\varepsilon_d^{-1}\left(\frac{c}{d}\right)$, where $(-)$ ia a modified quadratic symbol [6, p. 442, 447].

For $M\in\Gamma_0(q)$ and $Z\in H_n$, with $4|q$, we set

$$(1.3)\qquad j(M,Z)=\theta^n(M\langle Z\rangle)/\theta^n(Z).$$

Then the map $j:\Gamma_0(q)\to\widetilde{G}$ defined by $j(M)=(M,j(M,Z))$ is a well defined injective group homomorphism (see the Definition of $\widetilde{G}$ and (1.2)). We denote $j(\Gamma_0(q))$ by $\tilde{\Gamma}_0(q)$ and $j(M)$ by $\widetilde{M}$.

2. Siegel's modular forms of half integral weight

Let n, q be positive integers ≥ 1. Let k be a half integer >0 (i.e., $k=m/2$ for some positive integer m). Let $\zeta=(M,\alpha(Z))\in\widetilde{G}$. For an arbitrary function $F: H_n\to\boldsymbol{C}$, we set

$$(2.1)\quad (F|_k\zeta)(Z)=\det M^{(k-n-1)/2}\cdot\alpha(Z)^{-2k}\cdot F(M\langle Z\rangle),\quad Z\in H_n.$$

Since the map: $Z\to M\langle Z\rangle$ is an analytic automorphism on H_n and $\alpha(Z)\neq 0$ on H_n, $F|_k\zeta$ is holomorphic on H_n if F is. And from (2.1) it follows that $F|_k\zeta_1|_k\zeta_2=F|_k\zeta_1\zeta_2$ for $\zeta_1,\zeta_2\in\widetilde{G}$.

DEFINITION 2.1. Let n, q, k be as above and $4|q$. Let χ be a Dirichlet character modulo q. We define $M_k^n(q, \chi)$ to be the vector space over $\boldsymbol{C}$ spanned by functions $F: H_n \to \boldsymbol{C}$ such that

(i) F is holomorphic on H_n,

(ii) $F|_k\widetilde{M} = \chi(\det D_M)\cdot F$ for any $\widetilde{M} = (M, j(M, Z)) \in \widetilde{\Gamma}_0(q)$, and

(iii) if $n=1$, then $(cz+d)^{-k}\cdot F((az+b)(cz+d)^{-1})$ is bounded for every $(a, b; c, d) \in SL_2(\boldsymbol{Z})$ as $\operatorname{Im}(z) \to \infty$.

REMARK 2.2. If k is a positive integer, then $M_k^n(q, \chi)$ is the vector space over $\boldsymbol{C}$ spanned by complex valued functions F on H_n satisfying (i), (iii), and (ii)′ $F|_kM = \chi(\det D_M)\cdot F$ where the action of G on F is defined by

$$(2.2)\quad (F|_kM)(Z) = \det M^{k-(n+1)/2}\cdot \det(C_MZ+D_M)^{-k}\cdot F(M\langle Z\rangle),$$

$Z \in H_n$, for $M \in \Gamma_0(q)$.

REMARK 2.3. $M_k^n(q, \chi)$ ia a finite dimensional vector space over $\boldsymbol{C}$ in each case. From the conditions (ii) and (ii)′, it is easy to see that if n is odd, then $\chi(-1)=1$ when k is a half integer and $\chi(-1)=(-1)^k$ when k is an integer.

3. Theta-series of half integral weight

Let Q be a positive definite integral quadratic form in m variables, i.e., $Q(x) = \sum a_{ij}x_ix_j$, $1 \leq i, j \leq m$, a_{ii}, $2a_{ij} \in Z$. By Q we also denote the martix $(a_{ij}) \in M_m(\boldsymbol{Z}/2)$, which is a positive definite (eigenvalues>0), semi-integral (a_{ii}, $2a_{ij} \in \boldsymbol{Z}$), symmetric $m\times m$ matrix. We set for integer n, $1 \leq n \leq m$,

$$(3.1)\qquad \theta^n(Z, Q) = \sum e(Q[X]Z),\quad Z \in H_n,$$

where the summation is over $X \in M_{m,n}(\boldsymbol{Z})$. We call it the theta-series of degreen n associated to Q.

Let q be the smallest positive integer such that $q(2Q)^{-1}$ is integral with even diagonal entries and call it the level of Q.

Andrianov and Maloletkin [2, p. 230] proved the following:

THEOREM 3.1. *Let Q be a positive definite integral quadratic form in m variables, m even, Then*

$$(3.2)\qquad \theta^n(Z, Q) \in M_k^n(q, \chi),$$

where $M_k^n(q, \chi)$ *is the space introduced in Remark* 2.2 *with* $k=m/2\in\boldsymbol{Z}$ *and* $\chi=\chi_Q$ *is a Dirichlet character modulo* q *defined by* $\chi(\det D_M)=1$ or $(\mathrm{sign}(\det D_M))^k\cdot\left(\frac{(-1)^k \det 2Q}{|\det D_M|}\right)_{\mathrm{jac}}$, *respectively, for* $q=1$ *or* $q>1$, *where* $M\in\Gamma_0(q)$. *Here* $(-)_{\mathrm{jac}}$ *is the Jacobi symbol.*

We now extend Theorem 3.1 to the case when m is odd. Let Q be a positive definite integral quadratic form in m variables, m odd. It is known [3, 7] that the level q of Q in this case is divisible by 4. Let $Q^*=\mathrm{diag}(Q, E_3)$. Then Q^* is a positive definite integral quadratic form in $m+3$ variables where the level q^* of Q^* is same as the level q of Q. From (3.1) and the definition of the standard function it follows that

$$\theta^n(Z, Q^*)=(\theta^n(Z, Q)/\theta^n(Z))\cdot\theta^n(Z)^4,\quad Z\in H_n. \tag{3.3}$$

Since $m+3$ is even, we can apply Theorem 3.1 to Q^* to get

$$\theta^n(Z, Q^*)\,|_k M=\chi^*(\det D_M)\cdot\theta^n(Z, Q^*) \tag{3.4}$$

for and $M\in\Gamma_0(q)$ with

$$\chi^*(\det D_M)=(\mathrm{sign}(\det D_M))^{k^*}\cdot\left(\frac{(-1)^{k^*}\cdot\det 2Q^*}{|\det D_M|}\right)_{\mathrm{jac}} \tag{3.5}$$

where $\chi^*=\chi_{Q^*}$ and $k^*=(m+3)/2$ (see Remark 2.2).

From (2.2), (3.4) becomes

$$\theta^n(M\langle Z\rangle, Q^*)=\chi^*(\det D_M)\cdot\det(C_M Z+D_M)^k\cdot\theta^n(Z, Q^*). \tag{3.6}$$

Combining (3.3), (3.6), and the fact that $j(M, Z)^4=\det(C_M Z+D_M)^2$ (see (1.2), (1.3), and Remark 1.2), we obtain

$$\begin{aligned}\theta^n(M\langle Z\rangle, Q)=&\chi^*(\det D_M)\cdot\det(C_M Z+D_M)^{(m-1)/2}\\ &\cdot j(M, Z)\cdot\theta^n(Z, Q).\end{aligned} \tag{3.7}$$

REMARK 3.2. Stark [7] obtained (3.7) by setting $Q^*=\mathrm{diag}(Q, E_7)$. But as we hawe seen above, it is enough to take $Q^*=\mathrm{diag}(Q, E_3)$. He then suggested that one can complete the transformation formula by calculating $j(M, Z)$ for all $M\in\Gamma_0(q)$.

From (1.2), (2.1), and (3.7), it follows that for any $M\in\Gamma_0(q)$

$$\begin{aligned}\theta^n(Z,Q)\,|_k\widetilde{M}&=j(M, Z)^{-m}\cdot\theta^n(M\langle Z\rangle, Q)\\ &=\chi^*(\det D_M)\cdot\chi_\theta(M)^{-m+1}\cdot\theta^n(Z, Q),\end{aligned} \tag{3.8}$$

where $\widetilde{M}=(M, j(M,Z))\in\widetilde{\Gamma}_0(q)$, $4|q$.

Since $\theta^n(Z)^2=\theta^n(Z,E_2)$ and $-m+1$ is even, it immediately follows from Theorem 3.1 that

$$\text{(3.9)}\quad \chi_\theta(M)^{-m+1}=\chi_2(\det D_M)^{(-m+1)/2} = \left[\operatorname{sign}(\det D_M)\cdot\left(\frac{-1}{|\det D_M|}\right)_{\text{jac}}\right]^{(-m+1)/2}.$$

where $\chi_2=\chi_{E_2}$.

From (3.5) and (3.9), (3.8) becomes

$$\text{(3.10)}\qquad \theta^n(Z,Q)|_k\widetilde{M}=\chi_Q(\det D_M)\cdot\theta^n(Z,Q),$$

where

$$\text{(3.11)}\quad \chi_Q(\det D_M)=\chi^*(\det D_M)\cdot\chi_\theta(\det D_M)^{-m+1}=\left(\frac{2\det 2Q}{|\det D_M|}\right)_{\text{jac}}$$

for $M\in\Gamma_0(q)$, $4|q$. Note that $\chi_Q(-1)=1$.

This proves the condition (ii) of Definition 2.1 for $\theta^n(Z,Q)$. Since the conditions (i) and (iii) also hold for $\theta^n(Z,Q)$ (see [4, p. 18], for instance), we have proved the following generalization of Theorem 3.1 when m is odd:

THEOREM 3.3. *Let Q be a positive definite integral quadratic form in m variables, m odd. Let n be a positive integer and q be the level of Q, whence* $4|q$. Then

$$\theta^n(Z,Q)\in M_k^n(q,\chi)$$

where $M_k^n(q,\chi)$ is the space in Definition 2.1, $k=m/2$, and $\chi=\chi_Q$ *in* (3.11).

REMARK 3.4. For positive definite integral quadratic form Q in m variables, m odd, Carlsson and Johannßen [3, Satz 3.7] proved $\theta^n(M\langle Z\rangle,Q)=\chi_{c-j}(M)\cdot\det(C_MZ+D_M)^k\cdot\theta^n(Z,Q)$ with an explicit formula for $\chi_{c-j}(M)$ for any $M\in\Gamma_0(q)$, where q is the level of Q, $4|q$. The formula for $\chi_{c-j}(M)$, however, requires a reduction process due to Andrianov-Maloletkin [2, p. 240] of M to a matrix of the form

$$\begin{bmatrix} E_{n-1} & 0 & 0_{n-1} & 0 \\ 0 & \alpha & 0 & \beta \\ 0_{n-1} & 0 & E_{n-1} & 0 \\ 0 & \gamma & 0 & \delta \end{bmatrix}$$

so that $\chi_{c-j}(M)=\left(\frac{\det 2Q}{\delta}\right)_{\text{jac}}\cdot\left(\frac{2\gamma}{\delta}\right)_{\text{jac}}\cdot\varepsilon_\delta^{-m}$. Independently from Carlsson and Johannßen, Stark [7] found an explicit formula for $j(M,Z)$ when $M\in\Gamma_0(q)$ such that C_M^{-1} exists and D_M is of odd prime level, and suggested that one could obtain a transformation formula by extending his formula for $j(M,Z)$ to all $M\in\Gamma_0(q)$ (see Remark 3.2). Recently, Styer [8, Theorem 4] succeeded in generalizing the formula for $j(M,Z)$ to all $M\in\Gamma_0(q)$ by developing an analogue of Dirichlet' theorem for coprime symmetric pairs (C_M, D_M), and thereby he could give a transformation formula for $\theta^n(Z,Q)$ as Stark suggested. Although Styer's formula does not require the Andrianov-Maloletkin's reduction processs of M, it does require one to find an integral symmetric matrix S for a given coprime symmetric pair (C,D) such that $CS+D$ has an odd prime determinant. In fact, the character χ_Q of Theorem 3.3 is equal to $\chi_\theta^{-m}\cdot\chi_{c-j}$, of which product is simple as we saw in (3.11), while the individual factors χ_θ and χ_{c-j} are very difficult to calculate. As applications of Theorem 3.3, see [5] and [10] for instance.

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