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A SUBCLASS OF ANALYTIC p-VALENT FUNCTIONS

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1. Introduction

A function $f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}$ $(p\in N)$ which is analytic and p-valent in the unit disk $U=\{z:|z|<1\}$ is in the class $L_p(\alpha,\beta,\gamma)$ if it satisfies the condition

$$(1)\qquad \left|\frac{f'(z)}{pz^{p-1}}-1\right|<\beta\left|\alpha\frac{f'(z)}{pz^{p-1}}+(1-\gamma)\right|\quad(z\in U)$$

for some $\alpha(0\leqq\alpha\leqq1)$, $\beta(0<\beta\leqq1)$ and $\gamma(0\leqq\gamma<1)$.

In particular, the classes $L_1(0,\beta,0)$ and $L_1(1,\beta,0)$ were studied by Singh [7] and Padamanabhan [4], respectively. Recently, the class $L_1(\alpha,\beta,\gamma)$ were studied by Kim and Lee [1].

It is the purpose of this paper to get coefficient estimates and the radius of convexity for the class $L_p(\alpha,\beta,\gamma)$. Also we generalize some result of [1], [4] and [7].

2. Coefficient estimates

In the first place, we require the following lemma.

LEMMA 2.1. *Let a function*

$$H(z)=1+b_1z+b_2z^2+\cdots$$

be analytic in the unit disk U. Then *$H(z)$ is analytic and satisfies the condition*

$$\left|\frac{1-H(z)}{(1-\gamma)+\alpha H(z)}\right|<\beta$$

for $\alpha(0\leqq\alpha\leqq1)$, $\beta(0<\beta\leqq1)$ and $\gamma(0\leqq\gamma<1)$ if and only if there exists an analytic function $\phi(z)$ in the unit disk U such that $|\phi(z)|\leqq1$ for

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$z\in U$ and

$$H(z)=\frac{1-\beta(1-\gamma)z\phi(z)}{1+\alpha\beta z\phi(z)}.$$

Proof. Let a function

$$H(z)=1+b_1z+b_2z^2+\cdots$$

satisfies the condition (1). Setting

$$h(z)=\frac{1-H(z)}{\beta(1-\gamma)+\alpha\beta H(z)},$$

we note that the function $h(z)$ is analytic in the unit disk U, satisfies $|h(z)|<1$ for $z\in U$ and $h(0)=0$. Consequently, by using Schwarz's lemma, we have $h(z)=z\phi(z)$, where $\phi(z)$ is analytic in the unit disk U and satisfies $|\phi(z)|\leqq 1$ for $z\in U$. Thus we obtain

$$H(z)=\frac{1-\beta(1-\gamma)h(z)}{1+\alpha\beta h(z)}$$
$$=\frac{1-\beta(1-\gamma)z\phi(z)}{1+\alpha\beta z\phi(z)}.$$

On the other hand, if

$$H(z)=\frac{1-\beta(1-\gamma)z\phi(z)}{1+\alpha\beta z\phi(z)}$$

and $|\phi(z)|\leqq 1$ for $z\in U$, then the function $H(z)$ is analytic in the unit disk U. Furthermore, since $|z\phi(z)|\leqq|z|<1$ for $z\in U$, we get

$$\left|\frac{1-H(z)}{(1-\gamma)+\alpha H(z)}\right|=\beta|z\phi(z)|<\beta$$

for $z\in U$. This completes the proof of the lemma.

THEOREM 2.2. *If the function*

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}$$

is in the class $L_p(\alpha,\beta,\gamma)$. Then we have

$$|a_{p+n}|\leqq\frac{p\beta(1+\alpha-\gamma)}{p+n}$$

for any $n\geqq 1$. The result is sharp.

Proof. Since $f(z)$ is in the class $L_p(\alpha,\beta,\gamma)$, by Lemma 2.1, we have

$$\frac{f'(z)}{pz^{p-1}}=\frac{1-\beta(1-\gamma)h(z)}{1+\alpha\beta h(z)},$$

where $|h(z)|<1$ for $z\in U$ and $h(0)=0$. Consequently we obtain

$$((\gamma-1)pz^{p-1}-\alpha f'(z))\beta h(z)=f'(z)-pz^{p-1},$$

that is,

$$((\gamma-1-\alpha)pz^{p-1}-\sum_{n=1}^{\infty}\alpha(p+n)a_{p+n}z^{p+n-1})\beta h(z)$$
$$=\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n-1}.$$

Hence we can write

$$((\gamma-1-\alpha)pz^{p-1}-\sum_{n=1}^{k}\alpha(p+n)a_{p+n}z^{p+n-1})\beta h(z)$$
$$=\sum_{n=1}^{k+1}(p+n)a_{p+n}z^{p+n-1}+\sum_{n=k+1}^{\infty}c_n z^n,$$

c_n being some complex number. Now, $h(z)$ has modulus at most one in the unit disk U. Therefore, by using Parseval's identity, we get

$$\sum_{n=1}^{k+1}(p+n)^2|a_{p+n}|^2|z|^{2(p+n-1)}+\sum_{n=k+1}^{\infty}|c_n|^2|z|^{2n}$$
$$\leqq\beta^2(\gamma-1-\alpha)^2p^2|z|^{2(p-1)}+\alpha^2\beta^2\sum_{n=1}^{k}(p+n)^2|a_{p+n}|^2|z|^{2(p+n-1)}$$

Hence

$$\sum_{n=1}^{k+1}(p+n)^2|a_{p+n}|^2\leqq\beta^2p^2(1+\alpha-\gamma)^2+\alpha^2\beta^2\sum_{n=1}^{k}(p+n)^2|a_{p+n}|^2$$

and thus

$$(p+k+1)^2|a_{p+k+1}|^2\leqq\beta^2p^2(1+\alpha-\gamma)^2$$
$$-(1-\alpha^2\beta^2)\sum_{n=1}^{k}(p+n)^2|a_{p+n}|^2$$
$$\leqq\beta^2p^2(1+\alpha-\gamma)^2$$

because $0\leqq\alpha\leqq1$ and $0<\beta\leqq1$. This gives that

$$|a_{p+n}|\leqq\frac{p\beta(1+\alpha-\gamma)}{p+n}$$

for any $n\geqq1$. By taking

$$f(z)=\int_0^z pt^{p-1}\frac{1-\beta(\gamma-1)t^n}{1-\alpha\beta t^n}dt,$$

we have the condition (1) and the expression

$$f(z)=z^p+\frac{p\beta(1+\alpha-\gamma)}{p+n}z^{p+n}+\cdots$$

showing that the result is sharp.

3. A radius of convexity

THEOREM 3.1. *If the function*

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}$$

is in the class $L_p(\alpha,\beta,\gamma)$, *then the function* $f(z)$ *maps*

$$|z|<\frac{A-\sqrt{A^2-4p^2\beta(1-\gamma)}}{2\beta p(1-\gamma)}$$

onto a convex domain, where $A=p+\alpha\beta+\beta(1-\gamma)(1+p)$.

Proof. Let $f(z)$ belong to the class $L_p(\alpha,\beta,\gamma)$. Then we have

$$f'(z)=pz^{p-1}\frac{1-\beta(1-\gamma)z\phi(z)}{1+\alpha\beta z\phi(z)},$$

where $\phi(z)$ is analytic in the unit disk U and satisfies $|\phi(z)|\leqq 1$ for $z\in U$. Then we have

$$1+\frac{zf''(z)}{f'(z)}=p-\frac{\beta(1+\alpha-\gamma)(z\phi(z)+z^2\phi'(z))}{(1-\beta(1-\gamma)z\phi(z))(1+\alpha\beta z\phi(z))}.$$

Now, it is well known that

$$|\phi'(z)|\leqq\frac{1-|\phi(z)|^2}{1-|z|^2}$$

for the analytic function $\phi(z)$ in the unit disk U. Consequently, we have

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\geqq p-\frac{\beta((1+\alpha-\gamma)|z|(|\phi(z)|+|z\phi'(z)|)}{|(1-\beta(1-\gamma)z\phi(z))(1+\alpha\beta z\phi(z))|}$$

$$\geqq p-\frac{\beta(1+\alpha-\gamma)|z|(|z|+|\phi(z)|)(1-|z\phi(z)|)}{(1-|z|^2)|(1-\beta(1-\gamma)z\phi(z))(1+\alpha\beta z\phi(z))|}.$$

Furthermore, since

$$|(1+\alpha\beta z\phi(z))(1-\beta(1-\gamma)z\phi(z))|$$
$$\geqq(1-\alpha\beta|z\phi(z)|)(1-\beta(1-\gamma)|z\phi(z)|),$$

we get

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}$$
$$\geqq p-\frac{\beta(1+\alpha-\gamma)|z|(|z|+|\phi(z)|)(1-|z\phi(z)|)}{(1-|z|^2)(1-\alpha\beta|z\phi(z)|)(1-\beta(1-\gamma)|z\phi(z)|)}$$

$$\geqq p-\frac{\beta(1+\alpha-\gamma)|z|}{(1-|z|)(1-\beta(1-\gamma)|z|)}.$$

Hence if

$$|z|<\frac{A-\sqrt{A^2-4p^2\beta(1-\gamma)}}{2\beta p(1-\gamma)},$$

then we have

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0$$

where $A=p+\alpha\beta+\beta(1-\gamma)(1+p)$. Thus we have the theorm.

4. The class $L_p(\alpha,\beta,\gamma)$ with negative coefficients

Now, we generalize some results of [1], [4] and [7].

THEOREM 4.1. *A function* $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$ *is in the class* $L_p(\alpha,\beta,\gamma)$ *if and only if*

(2) $$\sum_{n=1}^{\infty}(1+\alpha\beta)(p+n)|a_{p+n}|\leqq p\beta(1+\alpha-\gamma).$$

Proof. Let $|z|=1$. Then

$$\begin{aligned}
&\left|\frac{f'(z)}{pz^{p-1}}-1\right|-\beta\left|\alpha\frac{f'(z)}{pz^{p-1}}+(1-\gamma)\right| \\
&\quad=\left|\sum_{n=1}^{\infty}\frac{p+n}{p}|a_{p+n}|z^n\right| \\
&\qquad-\beta\left|\alpha-\sum_{n=1}^{\infty}\alpha\frac{p+n}{p}|a_{p+n}|z^n+(1-\gamma)\right| \\
&\quad\leqq\sum_{n=1}^{\infty}\frac{p+n}{p}|a_{p+n}||z|^n \\
&\qquad-\beta(1+\alpha-\gamma)+\sum_{n=1}^{\infty}\beta\alpha\frac{p+n}{p}|a_{p+n}||z|^n \\
&\quad\leqq\sum_{n=1}^{\infty}(1+\alpha\beta)\frac{p+n}{p}|a_{p+n}|-\beta(1+\alpha-\gamma) \\
&\quad\leqq 0.
\end{aligned}$$

Thus we have that $f(z)$ is in $L_p(\alpha,\beta,\gamma)$.

Conversely, assume that $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$ is in $L_p(\alpha,\beta,\gamma)$. Then

$$\left|\frac{\frac{f'(z)}{pz^{p-1}}-1}{\alpha\frac{f'(z)}{pz^{p-1}}+(1-\gamma)}\right|<\beta.$$

Thus

$$\text{(3)}\qquad \mathrm{Re}\left\{\frac{\sum_{1}^{\infty}\frac{p+n}{p}|a_{p+n}|z^{n}}{(1+\alpha-\gamma)-\sum_{n=1}^{\infty}\alpha\frac{p+n}{p}|a_{p+n}|z^{n}}\right\}<\beta$$

Choose values of z on the real axis so that $f'(z)$ is real. Upon clearing the denominator in (3) and letting $z\to1$ through real values,

$$\sum_{n=1}^{\infty}(1+\alpha\beta)(p+n)|a_{p+n}|\leqq p\beta(1+\alpha-\gamma).$$

Letting $p=1$ in Theorem 4.1, we have the following.

COROLLARY 4.2. ([1]) *A function* $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$ *is in* $L_1(\alpha,\beta,\gamma)$ *if and only if*

$$\sum_{n=2}^{\infty}(1+\alpha\beta)n|a_n|\leqq\beta(1+\alpha-\gamma).$$

THEOREM 4.3. *If the function* $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$ *is in the class* $L_p(\alpha,\beta,\gamma)$, *then*

$$|z|^p-\frac{p\beta(1+\alpha-\gamma)}{(p+1)(1+\alpha\beta)}|z|^{p+1}\leqq|f(z)|\leqq|z|^p+\frac{p\beta(1+\alpha-\gamma)}{(p+1)(1+\alpha\beta)}|z|^{p+1}$$

and

$$p|z|^{p-1}-\frac{p(1+\alpha-\gamma)}{1+\alpha\beta}|z|^p\leqq|f'(z)|\leqq p|z|^{p-1}+\frac{p(1+\alpha-\gamma)}{1+\alpha\beta}|z|^p$$

for $z\in U$. *The results in sharp.*

Proof. Let $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$ be in the class $L_p(\alpha,\beta,\gamma)$. Then (2) gives

$$\sum_{n=1}^{\infty}|a_{p+n}|\leqq\frac{p\beta(1+\alpha-\gamma)}{(p+1)(1+\alpha\beta)}$$

Therefore, we have

$$|f(z)|\geqq|z|^p-|z|^{p+1}\sum_{n=1}^{\infty}|a_{p+n}|$$

$$\geqq |z|^p - \frac{p\beta(1+\alpha-\gamma)}{(p+1)(1+\alpha\beta)}|z|^{p+1}$$

and

$$|f(z)| \leqq |z|^p + |z|^{p+1}\sum_{n=1}^{\infty}|a_{p+n}|$$
$$\leqq |z|^p + \frac{p\beta(1+\alpha-\gamma)}{(p+1)(1+\alpha\beta)}|z|^{p+1}.$$

On the other hand, we have, from (2),

$$\sum_{n=1}^{\infty}(p+n)|a_{p+n}| \leqq \frac{p\beta(1+\alpha-\gamma)}{1+\alpha\beta}. \tag{4}$$

It follows from (4) that

$$|f'(z)| \geqq p|z|^{p-1} - |z|^p \sum_{n=1}^{\infty}(p+n)|a_{p+n}|$$
$$\geqq p|z|^{p-1} - \frac{p\beta(1+\alpha-\gamma)}{1+\alpha\beta}|z|^p.$$

and

$$|f'(z)| \leqq p|z|^{p-1} + |z|^p \sum_{n=1}^{\infty}(p+n)|a_{p+n}|$$
$$\leqq p|z|^{p-1} + \frac{p(1+\alpha-\gamma)}{1+\alpha\beta}|z|^p.$$

The result is sharp for the function

$$f(z) = z^p - \frac{p\beta(1+\alpha-\gamma)}{(p+1)(1+\alpha\beta)}z^{p+1}.$$

COROLLARY 4.4. *If the function* $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$ *is in the class* $L_p(\alpha,\beta,\gamma)$, *then the unit disk* U *is mapped on the domain which is contained in the disk with its center at the origin and radius* r *given by*

$$r = 1 + \frac{p\beta(1+\alpha-\gamma)}{(p+1)(1+\alpha\beta)}.$$

REMARK. For $p=1$, we obtain distortion theorems of Kim and Lee [1].

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