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# CERTAIN CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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## 1. Intorduction

Let $T_k$ be the class of functions of the form

$$f(z)=z-\sum_{n=k+1}^{\infty}a_nz^n \quad (a_n\geqq 0;\ k=1,2,3,\cdots) \tag{1.1}$$

which are analytic in the unit disk $U=\{z: |z|<1\}$.

A function $f(z)$ belonging to $T_k$ is said to be in the class $L_k(\alpha,\beta,\gamma)$ if and only if it satisfies the condition

$$\left|\frac{f'(z)-1}{\alpha f'(z)+(1-\gamma)}\right|<\beta \tag{1.2}$$

for some $\alpha(0\leqq\alpha\leqq 1)$, $\beta(0<\beta\leqq 1)$, $\gamma(0\leqq\gamma<1)$, and for all $z\in U$. The class $L_1(\alpha,\beta,\gamma)$ when $k=1$ were studied by Kim and Lee [1].

The object of the present paper is to prove some distortion inequalities for functions $f(z)$ belonging to the class $L_k(\alpha,\beta,\gamma)$. Furthermore, a new criterion for the class $L_k(\alpha,\beta,\gamma)$ is shown.

In order to derive our results for the class $L_k(\alpha,\beta,\gamma)$, we have to recall here the following lemma due to Kim and Lee [1].

LEMMA 1. *Let the function $f(z)$ be in the class $T_1$. Then $f(z)$ belongs to the class $L_1(\alpha,\beta,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty}(1+\alpha\beta)na_n\leqq\beta(\alpha+1-\gamma). \tag{1.3}$$

## 2. Distortion inequalities

Applying Lemma 1, we show

THEOREM 1. *Let the function $f(z)$ be in the class $T_k$. Then $f(z)$ belongs to the class $L_k$ $(\alpha,\beta,\gamma)$ if and only if*

$$\sum_{n=k+1}^{\infty}(1+\alpha\beta)na_n\leqq\beta(\alpha+1-\gamma). \tag{2.1}$$

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*Proof*. Letting $a_n=0$ for $n=2, 3, 4, \cdots, k$ in Lemma 1, we have the inequality (2.1). Further, the equality in (2.1) is attained for the function $f(z)$ given by

$$f(z)=z-\frac{\beta(\alpha+1-\gamma)}{n(1+\alpha\beta)}z^n \quad (n\geqq k+1). \tag{2.2}$$

COROLLARY 1. *If the function $f(z)$ of the form (1.1) is in the class $L_k(\alpha, \beta, \gamma)$, then*

$$a_n\leqq\frac{\beta(\alpha+1-\gamma)}{n(1+\alpha\beta)} \quad (n\geqq k+1). \tag{2.3}$$

*Equality in (2.3) is attained for the function $f(z)$ given by (2.2).*

With the help of Theorem 1, we prove

THEOREM 2. *If the function $f(z)$ of the form (1.1) is in the class $L_k(\alpha, \beta, \gamma)$, then*

$$|z|-\frac{\beta(\alpha+1-\gamma)}{(k+1)(1+\alpha\beta)}|z|^{k+1}\leqq|f(z)|\leqq|z| + \frac{\beta(\alpha+1-\gamma)}{(k+1)(1+\alpha\beta)}|z|^{k+1} \tag{2.4}$$

*and*

$$1-\frac{\beta(\alpha+1-\gamma)}{1+\alpha\beta}|z|^k\leqq|f'(z)|\leqq1+\frac{\beta(\alpha+1-\gamma)}{1+\alpha\beta}|z|^k \tag{2.5}$$

*for $z\in U$. The equalities in (2.4) and (2.5) are attained for the function $f(z)$ given by*

$$f(z)=z-\frac{\beta(\alpha+1-\gamma)}{(k+1)(1+\alpha\beta)}z^{k+1}. \tag{2.6}$$

*Proof*. Note that (2.1) gives

$$\sum_{n=k+1}^{\infty}a_n\leqq\frac{\beta(\alpha+1-\gamma)}{(k+1)(1+\alpha\beta)}. \tag{2.7}$$

Therefore, we have

$$\begin{aligned}|f(z)|&\geqq|z|-|z|^{k+1}\sum_{n=k+1}^{\infty}a_n\\&\geqq|z|-\frac{\beta(\alpha+1-\gamma)}{(k+1)(1+\alpha\beta)}|z|^{k+1}\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}|f(z)|&\leqq|z|+|z|^{k+1}\sum_{n=k+1}^{\infty}a_n\\&\leqq|z|+\frac{\beta(\alpha+1-\gamma)}{(k+1)(1+\alpha\beta)}|z|^{k+1}.\end{aligned} \tag{2.9}$$

On the other hand, we have, from (2.1),

$$\sum_{n=k+1}^{\infty} na_n \leqq \frac{\beta(\alpha+1-\gamma)}{1+\alpha\beta}. \tag{2.10}$$

It follows from (2.10) that

$$|f'(z)| \geqq 1-|z|^k \sum_{n=k+1}^{\infty} na_n \geqq 1-\frac{\beta(\alpha+1-\gamma)}{1+\alpha\beta}|z|^k \tag{2.11}$$

and

$$|f'(z)| \leqq 1+|z|^k \sum_{n=k+1}^{\infty} na_n \leqq 1+\frac{\beta(\alpha+1-\gamma)}{1+\alpha\beta}|z|^k. \tag{2.12}$$

COROLLARY 2. *If the function $f(z)$ of the form* (1.1) *is in the class $L_k(\alpha,\beta,\gamma)$, then the unit disk $U$ is mapped on the domain which is contained in the disk with its center at the origin and radius $r$ given by*

$$r=1+\frac{\beta(\alpha+1-\gamma)}{(k+1)(1+\alpha\beta)}. \tag{2.13}$$

## 3. Application to the fractional calculus

We need the following definitions of the fractional calculus (fractional integrals and fractional derivatives) by Owa ([3], [4]).

DEFINITION 1. The fractional integral of order $\delta$ is defined by

$$D_z^{-\delta}f(z)=\frac{1}{\Gamma(\delta)}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\delta}}d\zeta, \tag{3.1}$$

where $\delta>0$, $f(z)$ is an analytic function in a simply connected region of the $z$-plane containing the origin, and the multiplicity of $(z-\zeta)^{\delta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

DEFINITION 2. The fractional derivative of order $\delta$ is defined by

$$D_z^{\delta}f(z)=\frac{1}{\Gamma(1-\delta)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{\delta}}d\zeta \tag{3.2}$$

where $0\leqq\delta<1$, $f(z)$ is an analytic function in a simply connected region of the $z$-plane containing the origin and the multiplicity of $(z-\zeta)^{\delta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\delta)$ is defined by

(3.3) $$D_z^{n+\delta}f(z)=\frac{d^n}{dz^n}D_z^{\delta}f(z),$$

where $0\leqq\delta<1$ and $n=0,1,2,\cdots$.

By using the above definitions, we derive

THEOREM 3. *If the function $f(z)$ of the form (1.1) is in the class $L_k(\alpha,\beta,\gamma)$, then*

(3.4) $$|D_z^{-\delta}f(z)|\geqq\frac{|z|^{1-\delta}}{\Gamma(2+\delta)}\left\{1-\frac{\Gamma(k+1)\Gamma(2+\delta)\beta(\alpha+1-\gamma)}{\Gamma(k+2+\delta)(1+\alpha\beta)}|z|^k\right\}$$

*and*

(3.5) $$|D_z^{-\delta}f(z)|\leqq\frac{|z|^{1+\delta}}{\Gamma(2+\delta)}\left\{1+\frac{\Gamma(k+1)\Gamma(2+\delta)\beta(\alpha+1-\gamma)}{\Gamma(k+2+\delta)(1+\alpha\beta)}|z|^k\right\}$$

*for $\delta>0$ and $z\in U$. The equalities in (3.4) and (3.5) are attained for the function $f(z)$ given by (2.6).*

*Proof.* It is easy to see that Definition 1 gives

(3.6) $$\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z)=z-\sum_{n=k+1}^{\infty}\frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)}a_nz^n.$$

Letting

(3.7) $$\phi(n)=\frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)}\quad(n\geqq k+1),$$

we see that

(3.8) $$0<\phi(n)\leqq\phi(k+1)=\frac{\Gamma(k+2)\Gamma(2+\delta)}{\Gamma(k+2+\delta)}.$$

It follows from (2.7) and (3.8) that

(3.9) $$\begin{aligned}&|\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z)|\\&\geqq|z|-\phi(k+1)|z|^{k+1}\sum_{n=k+1}^{\infty}a_n\\&\geqq|z|-\frac{\Gamma(k+1)\Gamma(2+\delta)\beta(\alpha+1-\gamma)}{\Gamma(k+2+\delta)(1+\alpha\beta)}|z|^{k+1}\end{aligned}$$

which shows (3.4), and

(3.10) $$\begin{aligned}&|\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z)|\\&\leqq|z|+\phi(k+1)|z|^{k+1}\sum_{n=k+1}^{\infty}a_n\\&\leqq|z|+\frac{\Gamma(k+1)\Gamma(2+\delta)\beta(\alpha+1-\gamma)}{\Gamma(k+2+\delta)(1+\alpha\beta)}|z|^{k+1}\end{aligned}$$

which proves (3.5).

Finally, since the equalities in (3.4) and (3.5) are attained for the function $f(z)$ defined by

(3.11) $$D_z^{-\delta}f(z)=\frac{z^{1+\delta}}{\Gamma(2+\delta)}\left\{1-\frac{\Gamma(k+1)\Gamma(2+\delta)\beta(\alpha+1-\gamma)}{\Gamma(k+2+\delta)(1+\alpha\beta)}z^k\right\},$$

we know that the equalities in (3.4) and (3.5) are attained for the function $f(z)$ given by (2.6).

Next, we prove

THEOREM 4. *If the function* $f(z)$ *of the form* (1.1) *is in the class* $L_k(\alpha, \beta, \gamma)$, *then*

(3.12) $$|D_z^{\delta}f(z)|\geqq\frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left\{1-\frac{\Gamma(k+1)\Gamma(2-\delta)\beta(\alpha+1-\gamma)}{\Gamma(k+2-\delta)(1+\alpha\beta)}|z|^k\right\}$$

*and*

(3.13) $$|D_z^{\delta}f(z)|\leqq\frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left\{1+\frac{\Gamma(k+1)\Gamma(2-\delta)\beta(\alpha+1-\gamma)}{\Gamma(k+2-\delta)(1+\alpha\beta)}|z|^k\right\}$$

*for* $0\leqq\delta<1$ *and* $z\in U$. *The equalities in* (3.12) *and* (3.13) *are attained for the function* $f(z)$ *given by* (2.6).

*Proof*. Note that

(3.14) $$\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z)=z-\sum_{n=k+1}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}a_nz^n.$$

Defining the function $\phi(n)$ by

(3.15) $$\phi(n)=\frac{\Gamma(n)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}\quad(n\geqq k+1),$$

we can see that

(3.16) $$0<\phi(n)\leqq\phi(k+1)=\frac{\Gamma(k+1)\Gamma(2-\delta)}{\Gamma(k+2-\delta)}.$$

Therefore, by using (3.16) and (2.10), we have

(3.17) $$\begin{aligned}&|\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z)|\\&\geqq|z|-\phi(k+1)|z|^{k+1}\sum_{n=k+1}^{\infty}na_n\\&\geqq|z|-\frac{\Gamma(k+1)\Gamma(2-\delta)\beta(\alpha+1-\gamma)}{\Gamma(k+2-\delta)(1+\alpha\beta)}|z|^{k+1}\end{aligned}$$

which shows (3.12), and

(3.18) $$\begin{aligned}&|\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z)|\\&\leqq|z|+\phi(k+1)|z|^{k+1}\sum_{n=k+1}^{\infty}na_n\\&\leqq|z|+\frac{\Gamma(k+1)\Gamma(2-\delta)\beta(\alpha+1-\gamma)}{\Gamma(k+2-\delta)(1+\alpha\beta)}|z|^{k+1}\end{aligned}$$

which implies (3.13).

Furthermore, we note that the equalities in (3.12) and (3.13) are attained for the function $f(z)$ defined by

$$(3.19) \quad D_z^{\delta} f(z)=\frac{z^{1-\delta}}{\Gamma(2-\delta)}\left\{1-\frac{\Gamma(k+1)\Gamma(2-\delta)\beta(\alpha+1-\gamma)}{\Gamma(k+2-\delta)(1+\alpha\beta)} z^k\right\},$$

that is, defined by (2.6).

## 4. New criterion for $L_k(\alpha, \beta, \gamma)$

For the functions

$$(4.1) \qquad f_j(z)=z+\sum_{n=k+1}^{\infty} a_{n,j} z^n \quad (k=1,2,3,\cdots;\ j=1,2)$$

which are analytic in the unit disk $U$, we define the Hadamard product $f_1 * f_2(z)$ of $f_1$ and $f_2$ by

$$(4.2) \qquad f_1 * f_2(z)=z+\sum_{n=k+1}^{\infty} a_{n,1} a_{n,2} z^n.$$

We need the following result for analytic functions with negative coefficients.

LEMMA 2. *Let the function $f(z)$ of the form* (1.1) *be in the class $T_k$. Then $f(z)$ satisfies*

$$(4.3) \qquad Re\left\{\frac{f(z)}{z}\right\}>0$$

*if and only if*

$$(4.4) \qquad \sum_{n=k+1}^{\infty} a_n \leqq 1.$$

The above lemma owe to Sarangi and Uralegaddi [5].

With the aid of Lemma 2, we have

THEOREM 5. *If the function $f(z)$ of the form* (1.1) *is in the class $L_k(\alpha, \beta, \gamma)$, then $f*g(z)$ is close-to-convex in the unit disk $U$, where*

$$(4.5) \qquad g(z)=z+\sum_{n=k+1}^{\infty} \frac{1+\alpha\beta}{\beta(\alpha+1-\gamma)} z^n.$$

*Proof*. By using Theorem 1 and Lemma 2, we can see that

$$\begin{aligned} f(z) \in L_k(\alpha, \beta, \gamma) &\Longleftrightarrow \sum_{n=k+1}^{\infty} \frac{(1+\alpha\beta)n}{\beta(\alpha+1-\gamma)} a_n \leqq 1 \\ &\Longleftrightarrow \operatorname{Re}\left\{\frac{f*(zg'(z))}{z}\right\}>0 \\ &\Longleftrightarrow \operatorname{Re}\{(f*g(z))'\}>0. \end{aligned}$$

This implies that $f*g(z)$ is close-to-convex in the unit disk $U$.

COROLLARY 3. *If the function $f(z)$ of the form* (1.1) *is in the class*

$L_k(\alpha, \beta, \gamma)$, *then*

$$\text{(4.6)} \qquad Re\left\{\frac{f*g(z)}{z}\right\} > \frac{1}{3},$$

*where* $g(z)$ *is given by* (4.5).

*Proof*. By the result due to Obradović [2], we have

$$\text{(4.7)} \qquad \mathrm{Re}\{f'(z)\} > 0 \implies \mathrm{Re}\left\{\frac{f(z)}{z}\right\} > \frac{1}{3}$$

for an analytic function $f(z)$. It follows from the above and Theorem 5 that

$$\begin{aligned} f(z) \in L_k(\alpha, \beta, \gamma) &\iff \mathrm{Re}\{(f*g(z))'\} > 0 \\ &\implies \mathrm{Re}\left\{\frac{f*g(z)}{z}\right\} > \frac{1}{3} \end{aligned}$$

which completes the proof of Corollary 3.

## References


1. H.S. Kim and S.K. Lee, *Some classes of univalent functions*, to appear.
2. M. Obradović, *On certain inequalities for some regular functions in* $|z|<1$, Intern. J. Math. and Math. Sci. **8**(1985), 677–681.
3. S. Owa, *On the distortion theorems I*, Kyungpook Math. J. **18**(1978), 53–59.
4. S. Owa, *Some applications of fractional calculus*, Research Notes in Math. 138, Pitman, Boston, London and Melbourne, 1985, 164–175.
5. S.M. Sarangi and B.A. Uralegaddi, *The radius of convexity and starlikeness for certain classes of analytic functions with negative coefficients I*, Rend. Accad. Naz. Lincei **65**(1978), 38–42.


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