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REGULARITIES IN TOPOLOGICAL DYNAMICS

JUNG OK YU

0. Introduction

Given a transformation group (X, T) with a compact Hausdorff phase space X, we say points x and y of X are *proximal* provided, whenever α is a member of the unique compatible uniformity of X, there exists $t\in T$ such that $(xt, yt)\in\alpha$. In [12], this author has introduced the regular relations which are closely related to the concepts of regular minimal sets. These relations are defined to be the generalizations of the proximal relation. The points x and y of X are *regular* if $h(x)$ and y are proximal for some automorphism h of X. These relations are reflexive, symmetric and invariant, but is not in general transitive or closed.

In this paper we discuss the properties of regular relations more generally and a necessary and sufficient condition is given for these relations to be transitive.

1. Preliminaries

An arbitrary, but fixed, topological group will be denoted by T throughout this paper and we will consider the transformation group (X, T) with a compact Hausdorff space X. The compact Hausdorff space X carries a natural uniformity $\mathcal{U}$ whose indices are the neighborhoods of the diagonal in $X\times X$.

DEFINITION 1.1. Let (X, T) be a transformation group.

1) Let $x\in X$. The *orbit* of x, denoted by xT, is the subset $\{xt\mid t\in T\}$ of X.

2) A nonempty subset A of X is called a *minimal* set if for every $x\in A$ the orbit xT is a dense subset of A. If X is itself minimal, we

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say it is a *minimal transformation group.*

DEFINITION 1.2. Let (X, T) and (Y, T) be transformation groups. If π is a continuous map from X to Y with $\pi(xt)=\pi(x)t$ $(x\in X,\ t\in T)$, then π is called a *homomorphism. Endomorphism* and *automorphism* are defined obviously. The set of all automorphisms of X is denoted by $A(X)$.

DEFINITION 1.3. The *enveloping semigroup* $E(X)$ of X defined to be the closure of T in X^X, providing X^X with its product topology. The *minimal right ideal* I is the nonempty subset of $E(X)$ with $IE(X)\subset I$, which contains no proper nonempty subset with the same property. Let u and v be two idempotents in $E(X)$. Then u and v to be equivalent $(u\sim v)$ if $uv=u$ and $vu=v$.
The algebraic properties of enveloping semigroup, and their connection with the recursive properties of the transformation group are studied in [7].

For each $x\in X$, the map $\theta_x : p \longrightarrow xp$ of $E(X)$ into X is a homomorphism, and its image is just the orbit closure $\overline{xT}$ of x.

Let $\pi : (X, T) \longrightarrow (Y, T)$ be an epimorphism. It is well-known that there exists a unique epimorphism $\phi : (E(X), T) \longrightarrow (E(Y), T)$ such that $\pi\theta_x=\theta_{\pi(x)}\phi$ for each $x\in X$. Moreover, if π is a homomorphism of (X, T) onto (X, T), then ϕ is the identity map of $E(X)$ onto $E(X)$

DEFINITION 1.4. The *proximal relation* $P(X)$ of (X, T) is defined to be $\cap\{\alpha T|\alpha\in\mathscr{U}\}$. (X, T) is said to be *distal* if $P(X)=\Delta(X)$, the diagonal of $X\times X$.

DEFINITION 1.5. A point $x\in X$ is said to be *almost periodic* under T if given $\alpha\in\mathscr{U}$, there exists a syndetic subset A of T such that $xA\subset x\alpha$. (X, T) is *pointwise almost periodic* if and only if each point of X is almost periodic.

REMARK 1.6. Let $x\in X$, and let M be a minimal set contained in $\overline{xT}$. Then there exists a minimal right ideal I in $E(X)$ such that $M=xI$. ([4]).

2. Regular relations

DEFINITION 2.1. Let H be a subset of $A(X)$, and let $R_H(X, T)$ or simply $R_H(X)$ be the set of all $(x, y) \in X \times X$ such that $h(x)$ and y are proximal for some $h \in H$. The regular relation $R(X, T)$ or simply $R(X)$ on (X, T) is defined to be the set $R_{A(X)}(X)$ i.e., $(x, y) \in R(X)$ if and only if $(h(x), y) \in P(X)$ for some $h \in A(X)$.

REMARK 2.2. $P(X) = R_{1_X}(X) \subset R_H(X) \subset R(X)$, for every subgroup H of $A(X)$.

The following theorem is quite analogue to that of Theorem 2.2.3 [12] and the proofs will be omitted.

THEOREM 2.3. *Let (X, T) be a transformation group and let H be a subgroup of $A(X)$. Then the following statements hold.*

1) *$R_H(X, T)$ is a reflexive, symmetric and invariant relation.*

2) *If $E(X)$ contains just one minimal right ideal, then $R_H(X)$ is an equivalence relation.*

THEOREM 2.4. *Let H and K be subgroups of $A(X)$ and $H \subset K$. If $R_H(X)$ is an equivalence relation, then so is $R_K(X)$.*

Proof. By Theorem 2.3.1, we need only to show that $R_K(X)$ is transitive. Let $(x, y) \in R_K(X)$ and $(y, z) \in R_K(X)$. There exist k_1 and k_2 in K such that $(k_1(x), y) \in P(X) \subset R_H(X)$ and $(k_2(y), z) \in P(X) \subset R_H(X)$. Since $(k_2k_1(x), k_2(y)) \in P(X) \subset R_H(X)$ and $R_H(X)$ is transitive, $(k_2k_1(x), z) \in R_H(X)$. By definition of $R_H(X)$, there exists an $h \in H$ such that $(hk_2k_1(x), z) \in P(X)$. $H \subset K$ implies $hk_1k_2 \in K$ and thus we obtain $(x, z) \in R_K(X)$.

It is well-known that $P(X, T)$ is transitive if and only if there is only one minimal right ideal in $E(X)$. Therefore we have

COROLLARY 2.5. *If $P(X)$ is transitive, then so is $R_H(X)$ for every subgroup H of $A(X)$.*

The following theorem gives a necessary and sufficient condition for $R_H(X)$ to be an equivalence relation.

THEOREM 2.6. *Let (X, T) be a transformation group and let H be a subgroup of $A(X)$. The following are equivalent;*

1) *$R_H(X)$ is an equivalence relation.*

2) *Let u be an idempotent of enveloping semigroup $E(X)$. Then $(xu, yu) \in R_H(X)$ for every $(x, y) \in R_H(X)$.*

Proof. 1) implies 2). Let $(x, y) \in R_H(X)$ and $u^2 = u \in E(X)$. Note that $(x, xu) \in R_H(X)$ for all $x \in X$. Since (x, xu), (x, y), (y, yu) are all in $R_H(X)$ and $R_H(X)$ is transitive, it follows that $(xu, yu) \in R_H(X)$.

2) implies 1). It suffices to show that $R_H(X)$ is transitive. Let $(x, y) \in R_H(X)$ and $(y, z) \in R_H(X)$. For a given idempotent u of a minimal right ideal I of $E(X)$, we have $(xu, yu) \in R_H(X)$ and $(yu, zu) \in R_H(X)$ by assumption. There exist h_1 and h_2 in H such that $(h_1(xu), yu) \in P(X)$ and $(h_2(yu), zu) \in P(X)$. Therefore there exist minimal right ideals I' and I'' in $E(X)$ such that $h_1(xu)p = yup$, $h_2(yu)q = zuq$ for all $p \in I'$ and $q \in I''$.
Let $u \sim u' \sim u''$ with idempotents $u' \in I'$ and $u'' \in I''$. Then we obtain

$$h_1(xu)u' = yuu', \quad h_2(yu)u'' = zuu''$$

and

$$h_1(x)u = h_1(x)uu' = h_1(xu)u' = yuu' = yu.$$

Since H is a group, $h_1h_2 \in H$ and it follows that

$$h_2h_1(x)u = h_2(yu) = h_2(y)u = h_2(y)uu'' = zuu'' = zu,$$

which shows that $h_2h_1(x)$ and z are proximal. Thus, we have $(x, z) \in R_H(X)$. The proof is completed.

COROLLARY 2.7. *Let H be a subgroup of $A(X)$. If $R_H(X)$ is closed, then $R_H(X)$ is transitive.*

Proof. Let $(x, y) \in R_H(X)$ and $u^2 = u \in E(X)$. Since $R_H(X)$ is invariant and closed, $(x, y)u = \lim (x, y)t_\alpha = \lim (xt_\alpha, yt_\alpha) = (xu, yu) \in R_H(X)$.

In the following two corollaries give us the necessary and sufficient conditions for $P(X)$ and $R(X)$ to be transitive.

COROLLARY 2.8. *Let (X, T) be a transformation group. The following are equivalent;*

1) *$P(X)$ is an equivalence relation.*

2) *Let $u^2 = u \in E(X)$. Then $(xu, yu) \in P(X)$ for all $(x, y) \in P(X)$.*

3) *Let $x \in X$ and let u_1 and u_2 be any equivalent idempotents of $E(X)$. Then $(xu_1, xu_2) \in P(X)$.*

Proof. 1) implies 2). It follows from Theorem 2.6. $(H = 1_X)$

2) implies 3). Note that $(xu_1, x) \in P(X)$ for every $x \in X$ and $u_1^2 = u_1 \in E(X)$. Therefore $(xu_1, xu_2) = (xu_1, x)u_2 = (xu_1u_2, xu_2) \in P(X)$.

3) implies 1). We need only to show that $P(X)$ is transitive. Let $(x, y) \in P(X)$ and $(y, z) \in P(X)$. There exist minmal right ideals I

and I' such hat $xp=yp$ and $yq=zq$ for all $p\in I$ and $q\in I'$. Let $u\sim u'$ with idempotents $u\in I$ and $u'\in I'$. Then $xu=yu$ and $yu'=zu'$.
By assumption, we have $(yu, yu')\in P(X)$. Since (yu, yu') is an almost periodic point of $(X\times X, T)$, we obtain $yu=yu'$. Similarly, we also have $zu=zu'$. Therefore $xu=yu=zu'=zu$, which implies that $(x, z)\in P(X)$.

COROLLARY 2.9. *Let (X, T) be a transformation group. The following are equivalent;*

1) *$R(X)$ is an equivaence relation.*

2) *Let $u^2=u\in E(X)$. Then $(xu, yu)\in R(X)$ for all $(x, y)\in R(X)$.*

3) *Let $x\in X$ and let u_1 and u_2 be any equivalent idempotents of $E(X)$. Then $(xu_1, xu_2)\in R(X)$.*

Proof. 1) $\Longrightarrow$ 2) follows from Theorem 2.6$(H=A(X))$, 2) $\Longrightarrow$ 3) is straightforward, because $(xu_1, x)\in R(X)$ for all $x\in X$. 3) $\Longrightarrow$ 1) is due to ([12], Theorem 2.2.6).

THEOREM 2.10. *Let $x\in X$ and let M_1 and M_2 be minimal sets contained in $\overline{xT}$. Then there are points $y_i\in M_i$ such that $(x, y_i)\in R(X)$ for $i=1, 2$.*
Moreover, if $R(X)$ is transitive, then (M_1, T) and (M_2, T) are isomorphic.

Proof. The proof. of the first part follows from Lemma 2[4] and Corollary 2.9. Now, suppose that $R(X)$ is transitive. Since M_1 and M_2 are minimal sets in $\overline{xT}$, there exist minima right ideals I_1 and I_2 such that $xI_1=M_1$ and $xI_2=M_2$.
By Corollary 2.9, $(xu_1, xu_2)\in R(X)$, where $u_1{}^2=u_1\in I_1$, $u_2{}^2=u_2\in I_2$ and $u_1\sim u_2$. Hence we get $(h(xu_1), xu_2)\in P(X)$ for some h in $A(X)$. Therefore $h(xu_1)p=xu_2p$ for some $p\in E(X)$. Since $xu_1p\in xI_1E(X)\subset xI_1=M_1$ and $xu_2p\in xI_2E(X)\subset xI_2=M_2$, we have $h(xu_1)p=xu_2p\in h(M_1)\cap M_2$.
Now from the fact that $h(M_1)$ and M_2 are minimal, we obtain $h(M_1)=M_2$. This shows that (M_1, T) and (M_2, T) are isomorphic.

REMARK 2.11. Let (X, T) be a transformation group and let S be a syndetic subgroup of T. Then $R_H(X, S)=R_H(X, T)$ for subset H of $A(X)$.

THEOREM 2.12. *Let (X, T) be a transformation group. Let $h\in A(X)$.*

Suppose that the set $\{(x, y)\in X\times X|h(x)$ *and* y *are proximal*$\}$ *is closed. Then* $R_H(X)$ *is transitive for subgroup* H *of* $A(X)$.

Proof. Let $S=\{(x, y)\in X\times X|h(x)$ and y are proximal$\}$ be closed. From Corollary 1. [4] and Corollary 2.5, it is enough to show that $P(X)$ is closed. Let $((x_\alpha, y_\alpha))$ be a net in $P(X)$ such that $((x_\alpha, y_\alpha))$ converges to (x, y). Then $((h^{-1}(x_\alpha), y_\alpha)$ is a net in S and converges to $(h^{-1}(x), y)$. Since S is closed, we obtain $(h^{-1}(x), y)\in S$, that is, x and y are proximal and therefore $P(X)$ is closed.

REMARK 2.13. Suppose that there exists a $x_0\in X$ such that x_0 is regular and distal from all other points of X. Then $R(X)=X\times X$.

REMARK 2.14. Suppose $R(X)=X\times X$ and let (Y, T) be minimal. Then every two minimal sets in $(X\times Y, T)$ are isomorphic.

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Hannam University
Taejon 300, Korea