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OPERATORS HAVING ANALYTIC SPECTRAL RESOLVENTS

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1. Introduction

Throughout this paper, X is an abstract Banach space over the field of complex numbers $\boldsymbol{C}$, T is an element of $B(X)$, T^* denotes the dual operator of T on the dual space X^*. For a set $S\subset X$, $S^\perp$ is the annihilator of S, $\bar{S}$ for the closure of S in an appropriate topology and ∂S for the boundary of S. If T is endowed with the single valued extension property (SVEP), then $\sigma(x, T)$ denotes the local spectrum for $x\in X$, and $X_T(S)=\{x\in X : \sigma(x, T)\subset S\}$. If M is a T-invariant subspace, we write $T|M$ for the restriction and T/M for the operator induced by T on the quotient space X/M. We use $\sigma(T)$ for the spectrum of T and $\rho(T)$ for its resolvent set, the symbol $\mathrm{Cov}[\sigma(T)]$ stands for the class of all finite open coverings of $\sigma(T)$. We write $\mathrm{AI}(T)$, $\mathrm{Inv}(T)$ for the Analytic invariant subspaces, invariant subspaces of X for T respectively. And the symbol $\mathrm{SM}(T)$ denotes the spectral maximal subspaces of X for T.

A decomposable operator, analytic invariant subspaces, and analytic spectral resolvent appear in this paper frequently, we begin with their definitions.

DEFINITION 1.1. An operator $T\in B(X)$ said to be decomposable if, for every finite system $\{G_1, G_2, \cdots\cdots, G_n\}$ of open subsets of $\boldsymbol{C}$ that cover $\sigma(T)$, there exist spectral maximal subspaces $Y_1, Y_2, \cdots\cdots, Y_n$ such that

$$(1)\qquad X=\sum_{i=1}^{n} Y_i,$$

$$(2)\qquad \sigma(T|Y_i)\subset G_i \quad (i=1, 2, \cdots, n).$$

DEFINITION 1.2. A T-invariant subspace Y of X is said to be analytic invariant if, for every X-valued analytic function defined on a region

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$D \subset \boldsymbol{C}$ such that

$$(\lambda - T)f(\lambda) \in Y \text{ for } \lambda \in D, \text{ then } f(\lambda) \in Y \text{ for } \lambda \in D.$$

DEFINION 1.3. E is said to be an analytic spectral resolvent (ASR) for T if

(i) $E : \mathscr{U} \to \mathrm{AI}(T)$, where $\mathscr{U}$ is the usual topology of $\boldsymbol{C}$,

(ii) $E(\phi) = \{0\}$,

(iii) for $\{G_i\}_{i=1}^{n} \in \mathrm{Cov}[\sigma(T)]$, $X = \sum_{i=1}^{n} E(G_i)$, and

(iv) $\sigma(T | E(G)) \subset \overline{G}$ for any $G \in \mathscr{U}$.

In the definition 1.3, if one replace $\mathrm{AI}(T)$ by $\mathrm{Inv}(T)$ then E is called the spectral resolvent [13].

It is shown through lengthy computations that if $T \in B(X)$ has a spectral resolvent then T is a decomposable operator ([13], p. 77, Theorem 11). Thus it is true that if T has an ASR then T is decomposable. But the later case the decomposability follows from the following results:

THEOREM 1.4. [10]. *For an operator T, the following are equivalent.*

(i) T is decomposable.

(ii) For every open set G in $\boldsymbol{C}$, there is a T-invariant subspace M such that $\sigma(T/M) \subset \overline{G}$ and $\sigma(T/M) \subset \boldsymbol{C} \backslash G$.

THEOREM 15. [15]. *Let $E : \mathscr{U} \to Inv(T)$ be a spectral resolvent for T, then $\sigma(T/E(G)) \subset \boldsymbol{C} \backslash G$ if and only if $E(G)$ is analytic invariant under T.*

2. Invariance of an analytic spectral revolvent

In the first part of this section, we will give an answere to the following question:

If E is an ASR of $T_1 \in B(X)$, when does it also be an ASR for another operator $T_2 \in B(X)$?

Before stating the result, we need a Lemma and a definition.

DEFINITION 2.1. [3]. We say that T_1 and T_2 are quasi-nilpotent equivalent ($T_1 \overset{q}{\sim} T_2$) if,

$$\lim_{n \to \infty} \| (T_1 - T_2)^{[n]} \|^{\frac{1}{n}} = 0 = \lim_{n \to \infty} \| (T_2 - T_1)^{[n]} \|^{\frac{1}{n}},$$

where

$$(T_1-T_2)^{[n]}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}T_1^kT_2^{n-k}.$$

LEMMA 2.2. [3]. *(i) If* $T_1\overset{q}{\sim}T_2$ *then* $\sigma(T_1)=\sigma(T_2)$.

(ii) If T_1 *has the SVEP and if* $T_1\overset{q}{\sim}T_2$, *then* T_2 *has the SVEP.*

(iii) If T_1 *is decomposable and* $T_1\overset{q}{\sim}T_2$, *then* T_2 *is also decomposable and* $X_{T_1}(F)=X_{T_2}(F)$ *for every closed* $F\subset \boldsymbol{C}$.

(iv) If T *has the SVEP and if* $T_1\overset{q}{\sim}T_2$, *then* $\sigma(x, T_1)=\sigma(x, T_2)$ *for every* $x\in X$.

THEOREM 2.3. *For* $T_1, T_2\in B(X)$, *let* E *be an ASR for* T_1. *If* $T_1\overset{q}{\sim}T_2$ *and if* $E(G)\in AI(T_1)\cap AI(T_2)$ *for each* $G\in\mathfrak{U}$, *then* E *is also an ASR for* T_2.

Proof. By Lemma 2.2, (i), we have $\sigma(T_1)=\sigma(T_2)$. For any $\{G_i\}_{i=1}^n\in \mathrm{Cov}[\sigma(T_1)]$, $\sum_{i=1}^{n}E(G_i)=X$ and $\sigma(T_1|E(G))\subset\overline{G}$ for any $G\in\mathfrak{U}$.

It remains to prove that $\sigma(T_1|E(G))=\sigma(T_2|E(G))$ for any $G\in\mathfrak{U}$. For any $x\in E(G)\in \mathrm{AI}(T_1)\cap \mathrm{AI}(T_2)$, we have

$$\sigma(x, T_1)=\sigma(x, T_1|E(G)),\quad \sigma(x, T_2)=\sigma(x, T_2|E(G)).$$

Since T_1 is decomposable so it has the SVEP, thus $\sigma(x, T_1)=\sigma(x, T_2)$ by Lemma 2.2, (iv). Therefore we have

$$\bigcup_{x\in E(G)}\sigma(x, T_1)=\bigcup_{x\in E(G)}\sigma(x, T_1|E(G))=\sigma(T_1|E(G)),\ \text{and}$$
$$\bigcup_{x\subset E(G)}\sigma(x, T_2)=\sigma(T_2|E(G)).$$

It follows that $\sigma(T_1|E(G))=\sigma(T_2|E(G))$, $G\in\mathfrak{U}$.

The condition $E(G)\in \mathrm{AI}(T_1)\cap \mathrm{AI}(T_2)$ for every $G\in\mathfrak{U}$ in Theorem 2.3. seems to be crucial, but the following example shows it is not so unreasonable.

EXAMPLE. Let T be a spectral operator with the spectral measure μ in the sense of *N*. Dunford. Thus it can be represented by $T=S+Q$, where S and Q are the scalar and radical part respectively, and $\sigma(T)=\sigma(S)$. Putting $T_1=T$, $T_2=S$, we have

$$\|(T_1-T_2)^{[n]}\|^{\frac{1}{n}}=\|Q_n\|^{\frac{1}{n}}=\|(T_2-T_1)^{[n]}\|^{\frac{1}{n}},$$

whence $T_1 \overset{q}{\sim} T_2$.

We define $E(G)=\mu(\overline{G})X$ for $G\in\mathfrak{U}$, then we have $\mu(\overline{G})X=X_{T_1}(\overline{G})$, $X_{T_1}(\overline{G})=X_{T_2}(\overline{G})$ ([3], p. 40, Theorem 2.1). Moreover, $X_{T_1}(\overline{G})$ is a spectral maximal space for T_1 since T_1 is decomposable, thus $E(G)=X_{T_1}(\overline{G})$ is analytically invariant under T_1. i.e. $E(G)\in \mathrm{SM}(T_1)\subset \mathrm{AI}(T_1)$, $\mathrm{AI}(T_1)\cap \mathrm{AI}(T_2)\neq\phi$. Furthermore, T_1 and T_2 are decomposable, thus $E:\mathfrak{U}\to \mathrm{AI}(T_1)\cap \mathrm{AI}(T_2)$ is an ASR for T_1 and T_2.

A decomposable operator is not a strongly decomposable operator, thus if T is decomposable, the restriction operator $T|E(G)$ and the quotient operator $T/E(G)$ are not decomposable in general even if $E(G)$ is a spectral maximal space for T. We will give the conditions under which these operators are decomposable for some fixed $G\in\mathfrak{U}$. To do this we need a Lemma.

LEMMA 2.4. [14]. *Let T be decomposable on a reflexive Banach space X. If Y reduces T, then $T|Y$ is decomposable.*

For an operator T with the disconnected spectrum, $G\in\mathfrak{U}$ is said to be disconnect the spectrum $\sigma(T)$ if.

$$G\cap\sigma(T)\neq\phi,\quad \sigma(T)\not\subset G \text{ and } \partial G\subset\sigma(T).$$

PROPOSITION 2.5. *Let X be a reflexive Banach space, let E be an ASR for T. If $G\in\mathfrak{U}$ disconnects the spectrum $\sigma(T)$, then both $T|E(G)$ and $T/E(G)$ are decomposable.*

Proof. By the assumption on G, $X=E(G)\oplus E(\overline{G}^C)$ ([5], p. 62, Lemma 12). Thus $E(G)$ reduces T so by Lemma 2.4, $T|E(G)$ is decomposable. T^* is decomposable since T is, X^* is also reflexive and

$$\begin{aligned} X^*=E(G)^*\oplus E(\overline{G}^C)^* &\equiv (X/E(\overline{G}^C))^*\oplus(X/E(G))^* \\ &\equiv E(\overline{G}^C)^\perp\oplus E(G)^\perp. \end{aligned}$$

A simple computation shows that both $E(G)^\perp$ and $E(\overline{G}^C)^\perp$ are invariant under T^*. Using again the Lemma 2.4, $T^*|E(G)^\perp$ is decomposable.

Now, by the indentification $T^*|E(G)^\perp\equiv(T/E(G))^*$, $T/E(G)$ is decomposable. The last conclusion follows from the fact that T is decomposable if and only if T^* is decomposable. ([10], p. 95, Corollary 1).

3. Constructions of new ASR from given ASR.

In this section, we will formulate an ASR for the functional calculus, the direct sum of two ASR and a transformation of an ASR by an invertible operator. In what follows, we use the symbol $\mathfrak{F}(T)$ for the set of all non-constant complex valued analytic functions on some open neighborhood of the spectrum $\sigma(T)$, and $f(T)$ for the functional calculus for T if $f\in\mathfrak{F}(T)$.

A T-invariant subspace Y of X is said to be a ν-space if $\sigma(T|Y)\subset\sigma(T)$.

LEMMA 3.1. [9]. *Let $T\in B(X)$, let g be analytic on some open neighborhood of $\sigma(T)$. If Y is analytically invariant under T, then Y is analytically invariant under $g(T)$.*

LEMMA 3.2. [6]. (1) *Given $T\in B(X)$, let f be an analytic injective function on some neighborhood of $\sigma(T)$. Then Y is a ν-space for $f(T)$ then Y is a ν-space for T.*

(2) *Given $T\in B(X)$, let f be an analytic function on an open neighborhood of $\sigma(T)$. If Y is ν-space for T then Y is ν-space for $f(T)$. Furthermore, we have*

$$f(T)|Y=f(T|Y),\quad f(T)/Y=f(T/Y).$$

(3) *Given $T\in B(X)$, let $f: D\to\boldsymbol{C}$ be analytic on an open neighborhood D of $\sigma(T)$ and nonconstant on every component of D.*
If $Y\in AI[f(T)]$ and $Y\in Inv(T)$, then $Y\in AI(T)$.

THEOREM 3.3. *Let E be an ASR for T, let $f\in\mathfrak{F}(T)$ and if f is continuous on $\boldsymbol{C}$ then the map $\mathcal{E}$ defined by $\mathcal{E}=E\circ f^{-1}$ is an ASR for $f(T)$.*

Proof. For any $\{H_i\}_{i=1}^{n}\in\text{Cov}[\sigma(f(T))]$, $f(\sigma(T))=\sigma(f(T))\subset\bigcup_{i=1}^{n}H_i$, whence

$$\sigma(T)\subset f^{-1}\left(\bigcup_{i=1}^{n}H_i\right)=\bigcup_{i=1}^{n}f^{-1}(H_i)$$

We put $f^{-1}(H_i)=G_i$ $(i=1,2,\cdots,n)$, then $\{G_i\}_i\in\text{Cov}[\sigma(T)]$. Since E is an ASR for T, we have

$$\mathcal{E}(\phi)=\{0\},\ \sum_{i=1}^{n}\mathcal{E}(H_i)=\sum_{i=1}^{n}E(G_i)=X,\ \text{and}$$

For each $H\in\mathcal{U}$, putting $f^{-1}(H)=G$, $\mathcal{E}(H)=E(G)$ is an analytically invarint subspace under $f(T)$ by Lemma 3.1. Thus $\mathcal{E}(H)$ is a ν-space for $f(T)$. Therefore, by Lemma 3.2, (2), we have

$$\sigma(f(T)\,|\,\mathcal{E}(H))=\sigma(f(T|\mathcal{E}(H)))=f(\sigma(T|\mathcal{E}(H)))=f(\sigma(T|E(G))).$$

And since $\sigma(T|E(G))\subset\overline{G}\cap\sigma(T)$,

$$\sigma(f(T)\,|\,\mathcal{E}(H))\subset f(\overline{G})\cap f(\sigma(T))\subset\overline{H}\cap\sigma(f(T))$$

hold by the continuity of f and the spectral mapping theorem. Therefore, $\mathcal{E}:\mathcal{U}\to\mathrm{AI}[f(T)]$ is an ASR for $f(T)$.

The converse of the Theorem 3.3 is following.

THEOREM 3.4. *Let $f\in\mathcal{F}(T)$, let f be an injective open function on $\boldsymbol{C}$. If $\mathcal{E}$ is an ASR for $f(T)$, then the map E defined by $E=\mathcal{E}\circ f$ is an ASR for T.*

Proof. Let $\{G_i\}_{i=1}^{n}\in\mathrm{Cov}[\sigma(T)]$. Then $\{f(G_i)\}_{i=1}^{n}\in\mathrm{Cov}\ [\sigma(f(T))]$ by the spectral mapping theorem, $\mathcal{E}(f(G_i))\in\mathrm{AI}(f(T))$ for each i. We put $f(G_i)=H_i$. Since $E(G_i)=\mathcal{E}(H_i)$, we have

$$E(\phi)=\mathcal{E}(\phi)=\{0\},\ \sum_i E(G_i)=\sum_i\mathcal{E}(H_i)=X.$$

For an admissible contour C which surrounds $\sigma(T)$ and is contained in $D\cap\rho(T)$, where D is a some open neighborhood of $\sigma(T)$. Applying Dunford's theorem on composite operator-valued function to the composite $f^{-1}\circ f$, we have

$$f^{-1}[f(T)]=\frac{1}{2\pi i}\int_C f^{-1}[f(\lambda)]R(\lambda,T)\,d\lambda=\frac{1}{2\pi i}\int_C\lambda R(\lambda,T)\,d\lambda=T.$$

By the same arguments as the proof of Theorem 3.3,

$$\begin{aligned}\sigma(T|E(G))&=\sigma(f^{-1}[f(T)]\,|\,E(G))=\sigma(f^{-1}[f(T)]\,|\,\mathcal{E}(H))\\&=\sigma(f^{-1}[f(T|\mathcal{E}(H))])=f^{-1}[\sigma(f(T)\,|\,\mathcal{E}(H))]\\&\subset f^{-1}[\overline{H}\cap\sigma(f(T))]=f^{-1}(\overline{H})\cap\sigma(T).\end{aligned}$$

And since f is an injective open function on $\boldsymbol{C}$,

$$f^{-1}(\overline{H})=f^{-1}(\overline{f(G)})\subset\overline{f^{-1}(f(G))}=\overline{G}.$$

Thus we have $\sigma(T|E(G))\subset\overline{G}\cap\sigma(T)$ for $G\in\mathcal{U}$, therefore $E:\mathcal{U}\to\mathrm{Inv}(T)$ is a spectral resolvent for T.

It remains to show that $E(G)=\mathcal{E}(H)$ is analytically invariant under T for each $G\in\mathcal{U}$, where $H=f(G)$.

Since $E(G)=\mathcal{E}(H)$ is analytically invariant under $f(T)$,

$$(3.a)\quad \sigma(f(T)/\mathcal{E}(H))\subset H^C\cap\sigma(f(T))=f(G)^C\cap f(\sigma(T))$$
$$=f(G^C\cap\sigma(T)),$$

the last equality holds since f is injective.

Now, by Lemma 3.2, $\mathcal{E}(H)=E(G)$ is a ν-space for $f(T)$ if and only if it is a ν-space for T. Thus

$$(3.b)\quad [f(T)/\mathcal{E}(H)]=\sigma[f(T/E(G))]=f(\sigma(T/E(G))).$$

It follows from (3.a) and (3.b) that

$$\sigma(T/E(G))\subset G^C\cap\sigma(T)\subset \boldsymbol{C}\backslash G.$$

Therefore, by Theorem 1.5, $E(G)$ is analytically invariant under T for each $G\in\mathfrak{U}$. This completes the proof.

From Theorem 3.3 and Theorem 3.4, we have the following corollary.

COROLLARY 3.5. *Let $f\in\mathfrak{F}(T)$, if $f:\boldsymbol{C}\to\boldsymbol{C}$ is a homeomorphism then E is an ASR for T if and only if $\mathcal{E}=E\circ f^{-1}$ is an ASR for $f(T)$.*

THEOREM 3.6. *Let X_k $(k=1,2)$ be Banach spaces, let $T_k\in B(X_k)$ $(k=1,2)$. If E_k is an ASR for T_k for $k=1,2$. Then the map E defined by $E(G)=E_1(G)\oplus E_2(G)$ $(G\in\mathfrak{U})$ is an ASR for $T=T_1\oplus T_2$.*

Proof. For $\{G_i\}_{i=1}^n\in\mathrm{Cov}[\sigma(T)]$, $\{G_i\}_{i=1}^n\in\mathrm{Cov}[\sigma(T_k)]$ $(k=1,2)$ since $\sigma(T)=\sigma(T_1)\cup\sigma(T_2)$. Thus $\sum_i E_k(G_i)=X_k$ $(k=1,2)$ and $\sum_i E(G_i)=X$. Clearly $E(\phi)=\{0\}$.

For any $G\in\mathfrak{U}$, $E(G)\in\mathrm{AI}(T)$ ([6], p. 20, proposition 2.18). Furthermore,

$$\sigma(T|E(G))=\sigma(T_1\oplus T_2|E_1(G)\oplus E_2(G))$$
$$=\sigma(T_1|E_1(G))\cup\sigma(T_2|E_2(G))\subset\overline{G}$$

Therefore, $E:\mathfrak{U}\to\mathrm{AI}(T)$ is an ASR for T.

For the converse of the Theorem 3.6, we need some basic results for analytically invariant subspaces.

LEMMA 3.7. [9]. (1) *Let $T_i\in B(X_i)$ $(i=1,2)$, let Y_i be an T_i-invariant subspace $(i=1,2)$. Then $Y_1\oplus Y_2$ is analytically invariant under $T_1\oplus T_2$ if and only if Y_i is analytically invariant under T_i.*

(2) *Let Y, Z be T-invariant subspaces with $Y\subset Z$. Then the following hold.*

(*i*) *If $Y\in AI(T)$, then $Y\in AI(T|Z)$.*

(*ii*) *If* $Y\in AI(T|Z)$ *and* $Z\in AI(T)$, *then* $Y\in AI(T)$.

(*iii*) $Z\in AI(T)$ *if and only if* $Z/Y\in AI(T/Y)$.

(*iv*) *If* T *have the SVEP and let* P *be a bounded projection operator on* X *commuting with* T, *then* $PX\in AI(T)$.

LEMMA 3.8. [6]. *Let* T *have the SVEP and let* $Y\in Inv(T)$. *Then* $T|Y$ *has the SVEP and* $\sigma(y, T)\subset\sigma(y, T|Y)$ for every $y\in Y$.

THEOREM 3.9. *Let* $T\in B(X)$, $X=X_1\oplus X_2$ *and let* P_1 *be a bounded projection operator of* X *onto* X_1 *commuting with* T.

If E *is an ASR for* T, *then the map* $E_k=P_k\circ E$ $(k=1,2)$ *are ASR for* T_k $(k=1,2)$, *where* $P_2=I-P_1$, $T_k=T|X_k$ *for* $k=1,2$.

Proof. We note that the condition $TP_1=P_1T$ is equivalent to $TX_1\subset X_1$, $TX_2\subset X_2$. Obviously, $E_k(G)=P_kE(G)$ is a closed subspace for $k=1,2$.

Since $TE(G)\subset E(G)$ $(G\in\mathfrak{U})$, $P_kTE(G)\subset P_kE(G)=E_k(G)$, we have $TE_k(G)\subset E_k(G)$. Hence $T_kE_k(G)=(T|X_k)E_k(G)=TE_k(G)\subset E_k(G)$.

Now we will show that $E_k(G)\in AI(T|E(G))$, $k=1,2$. Putting $A=T|E(G)$, $A: E(G)\to E(G)$ have the SVEP by Lemma 3.8. $P_kT=TP_k$ implies that $P_kA=AP_k$. Therefore $P_kE(G)=E_k(G)$ $(k=1,2)$ are considered as subspaces of $E(G)$ invariant under A, and $E_k(G)\in AI(A)=AI(T|E(G))$ by Lemma 3.7, (2), (ii) and (iv).

It follows that

$$\sigma(T_1|E_1(G))=\sigma(T|E_1(G))=\sigma([T|E(G)]|E_1(G))\subset\sigma(T|E(G)),$$

the last inclusion relation follows from Lemma 3.7, (i) and every analytically invariant subspace is a ν-space. Thus we have

$$\sigma(T_1|E_1(G))\subset\overline{G}, \text{ and similary } \sigma(T_2|E_2(G))\subset\overline{G}.$$

Obviously $E_k(\phi)=\{0\}$, $k=1,2$. It remains to prove that $\sum_{i=1}^{n}E_k(G_i)=X_k$ for every $\{G_i\}_{i=1}^{n}\in\mathrm{Cov}[\sigma(T_k)]$ $(k=1,2)$.

We choose $G\in\mathfrak{U}$ such that $\sigma(T_1)\cap\overline{G}=\phi$ and satisfying $\left(\bigcup_{i=1}^{n}G_i\right)\cup G\supset\sigma(T)$.

By Lemma 3.7, (1), $E_1(G)\oplus E_2(G)=E(G)\in AI(T)$ if and only if $E_k(G)\in AI(T_k)$ for $k=1,2$. Thus $\sigma(T_1|E_1(G))\subset\overline{G}\cap\sigma(T_1)=\phi$. Therefore,

$$\sigma(T_1)\cap\overline{G}=\phi \text{ implies that } E_1(G)=\{0\}.$$

Since $\{G_1, G_2, \cdots, G_n, G\}\in\mathrm{Cov}[\sigma(T)]$, $\sum_{i=1}^{n}E(G_i)+E(G)=X$. So we have $\sum_{i=1}^{n} E_1(G_i)=X_1$. Similary $\sum_{i=1}^{n} E_2(G_i)=X_2$ for any $\{G_i\}_{i=1}^{n}\in \mathrm{Cov}[\sigma(T_2)]$. We have proved the Theorem.

THEOREM 3.10. *Let E be an ASR for $T\in B(X)$, let Y be another Banach space. If T is similar to $S\in B(Y)$, then the map $\mathcal{E}=VE$ defined by $\mathcal{E}(G)=VE(G)$ $(G\in\mathfrak{U})$ is an ASR for S, where $V\in B(X,Y)$ is an invertible operator such that $VT=SV$.*

Proof. We propose to show that $\mathcal{E}(G)=VE(G)\in AI(S)$. Clearly, $VE(G)$ is a closed subspace of X for each $G\in\mathfrak{U}$, and invariant under S.

Let $f:D\to X$ be analytic and satisfy

$$(\lambda I-S)f(\lambda)\in VE(G) \text{ on } D.$$

Then

$$V^{-1}(\lambda I-S)f(\lambda)\in E(G) \text{ and } (\lambda I-T)V^{-1}f(\lambda)\in E(G) \text{ on } D.$$

And since $V^{-1}f(\lambda)$ is analytic on D and $E(G)\in AI(T)$, we have

$$V^{-1}f(\lambda)\in E(G), \text{ thus } f(\lambda)\in VE(G),\ \lambda\in D.$$

Now, for any $\{G_i\}_{i=1}^{n}\in\mathrm{Cov}[\sigma(T)]=\mathrm{Cov}[\sigma(S)]$,

$$\sum_{i=1}^{n} VE(G_i)=V\sum_{i=1}^{n} E(G_i)=VX=Y, \text{ i.e. } \sum_{i=1}^{n}\mathcal{E}(G_i)=Y.$$

Furthermore, $S|VE(G)$ is similar to $T|E(G)$; this follows from the facts that

$$[V|E(G)][T|E(G)]=[S|VE(G)][V|E(G)], \text{ and}$$

$V|E(G)$ is invertible. Theorefore,

$$\sigma(S|VE(G))=\sigma(T|E(G))\subset\bar{G}\cap\sigma(T) \text{ i.e. } \sigma(S|\mathcal{E}(G))\subset\bar{G}\cap\sigma(S).$$

Hence $\mathcal{E}:\mathfrak{U}\to \mathrm{AI}(S)$ is an ASR for S.

In difinition of the ASR E for T if $n=2$ then E is said to be two-ASR for T. Auther previously proved that if T has a two-ASR E, then the dual operator T^* has also a two-ASR E^*, where $E^*:\mathfrak{U}\to \mathrm{AI}(T^*)$ is defined by $E^*(G)=E(\boldsymbol{C}\backslash\bar{G})^{\perp}$ ([12], p. 77, Theorem 6).

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