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MAPPING THEOREMS FOR LOCALLY EXPANSIVE OPERATORS

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1. Introduction

It is well-known fact [2, p. 62] that if a local homeomorphism of a Banach space X into a Banach space Y is a local expansion, in the sense that for a continuous nonincreasing function $c:[0,\infty)\rightarrow(0,\infty)$ with $\int_0^\infty c(t)dt=\infty$, each point x of X has a neighborhood U_x such that

$$c(\max\{\|u\|,\|v\|\})\|u-v\|\leq\|Tu-Tv\|$$

for each u, v in U_x, then $T(X)=Y$.

Kirk and Schöneberg [3], and Ray and Walker [5] proved that a similar result can be obtained within the class of mappings whose graphs are closed subsets of $X\times Y$. Also Torrejon [6] obtained the same result without assuming that c is nonincreasing. Morever, Bae and Yie [1] proved a more stronger result by giving the precise range of the operator T, that is, they proved that under the same situation $T(B(0;K))$ contains $B(T(0);\int_0^K c(t)dt)$.

This note is a continuation of the above program; by developing Torrejon's method, we replace the domain of c by X instead of $[0,\infty)$ and give more general results which contain all the above mentioned results of [1, 2, 3, 5, 6].

First we give some notations and definitions.

If D is a subset of X, then $\bar{D}$ and ∂D denote, respectively, the closure and boundary of D in X. Recall that a mapping $T:D\longrightarrow Y$ is said to have *closed graph* if for each sequence $\{x_n\}\subseteq D$ with $x_n\longrightarrow x\in D$ and $Tx_n\longrightarrow y$ as $n\longrightarrow\infty$, it follows that $Tx=y$. We denote by $B(x,r)$ the set $\{y;\|y-x\|<r\}$, and conveniently we set $B(x;\infty)=X$ if $x\in X$.

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A continuous curve $h:[0,s]\longrightarrow X$, $0\leq s<\infty$, is *rectifiable* if there exists a constant $M>0$ such that for any subdivision of $[0,s]$ of the form

$$0=t_0<t_1<\cdots<t_n=s$$

we have

$$\sum_{i=1}^{n}\|h(t_i)-h(t_{i-1})\|\leq M.$$

The least such contant M is called the *length* of the curve. A continuous curve $h:[0,s]\longrightarrow X$ is said to be *parametrized by arc length* if for any t, $0\leqslant t\leqslant s$, the length of the curve $h|_{[0,t]}$ is exactly t.

REMARK 1. Note that if $h:[0,s)\longrightarrow X$ is a parametrized curve by arc length with $s<\infty$, then $\lim_{t\to s^-} h(t)$ always exists and h can be extended to $[0,s]$.

A nonlinear operator T mapping a subset D of X into a metric space Y is said to be *locally m-expansive*, where $m:D\longrightarrow(0,\infty)$ is a continuous function, if each point x in D has a neighborhood U_x such that

$$(*)\qquad \min\{m(u),\ m(v)\}\|u-v\|\leqslant d(Tu,Tv)$$

for each u,v in U_x.

Following Menger [4], a metric space Y is said to be *metrically convex* if for each x,y in Y with $x\neq y$ there exists z in Y, distinct from x and y, such that $d(x,y)=d(x,z)+d(z,y)$.

2. Main results

Now we state our first theorem.

THEOREM 1. *Let X be a Banach space, D an open subset of X, and let Y be a complete metric space with metric convexity. Let $m:D\longrightarrow(0,\infty)$ be a continuous function such that*

$$(**)\qquad \int_0^\infty m(h(t))dt=\infty \text{ for any continuous curve } h:[0,\infty)\longrightarrow D$$
parametrized by arc lengh.

Let $T:\bar{D}\longrightarrow Y$ be a locally m-expansive mapping on D having closed graph. If T maps open subsets of D onto open subsets of Y, then for each $y\in Y$ the followings are equivalent.

(1) $y \in T(D)$.

(2) *There exists* $x_0 \in D$ *such that* $d(Tx_0, y) \leqslant d(Tx, y)$ *for all* $x \in \partial D$.

Proof. We only need to prove that (2)$\longrightarrow$(1). We let $g: [0, d(Tx_0, y)] \longrightarrow Y$ be an isometry such that $g(0) = Tx_0$ and $g(d(Tx_0, y)) = y$. The existence of g is assured by Menger's result [4]. Let M denote the set of all τ in $[0, d(Tx_0, y)]$ for which there exists a unique continuous curve $h: [0, \tau] \longrightarrow D$ such that $h(0) = x_0$ and $Th(t) = g(t)$ for each t, $0 \leqslant t \leqslant \tau$. Let $\tau_0 = \sup\{\tau;\ \tau \in M\}$. Then $\tau_0 > 0$ since T is assumed to be an open and locally m-expansive mapping on D. Now we claim that $\tau_0 \in M$, so that we conclude that $\tau_0 = d(Tx_0, y)$ since M is an open subset of $[0, d(Tx_0, y)]$, and hence $Th(\tau_0) = y \in T(D)$. Since $\tau_0 = \sup\{\tau;\ \tau \in M\}$, there is a unique continuous curve $h: [0, \tau_0) \longrightarrow D$ with $h(0) = x_0$ and $Th(t) = g(t)$ for each t, $0 \leqslant t < \tau_0$.

LEMMA 1. *For each* $\tau \in [0, \tau_0)$, *the curve* $h|_{[0,\tau]}$ *is rectifiable.*

Proof. Since $[0, \tau]$ is compact, $\inf\{m(h(t));\ 0 \leqslant t \leqslant \tau\} = m > 0$. Then it is easily seen that for any subdivision

$$0 = t_0 < t_1 < \cdots < t_n = \tau$$

of $[0, \tau]$ we have

$$\sum_{i=1}^{n} \|h(t_i) - h(t_{i-1})\| \leqslant \frac{\tau}{m},$$

so that $h|_{[0,\tau]}$ is rectifiable.

Continuation of the proof of Theorem 1. By Lemma 1, for each t, $0 \leqslant t < \tau_0$, the length of the curve $h|_{[0,t]}$ exists, and we denote it by $s(t)$. Then note that $s: [0, \tau_0) \longrightarrow [0, \infty)$ is a continuous strictly increasing function. To complete our proof, we need another lemma.

LEMMA 2. *For each fixed* $t \in [0, \tau_0)$, *we have*

$$m(h(t))D^+ s(t) \leqslant 1,$$

where $D^+ v$ *is the right-upper Dini derivative of the function* v.

Proof. For any given $\varepsilon > 0$, we have a neighborhood $U_{h(t)}$ of $h(t)$ in D such that $m(x) \geqslant (1 - \varepsilon) m(h(t))$ for all $x \in U_{h(t)}$ and (*) holds. Now choose $r > 0$ such that for all t' with $t \leqslant t' \leqslant t + r < \tau_0$, $h(t') \in U_{h(t)}$. Also we can choose a subdivision

$$t=t_0<t_1<\cdots<t_n=t+r$$

of $[t, t+r]$ such that

$$s(t+r)-s(t)\leqslant(1+\varepsilon)\sum_{i=1}^{n}\|h(t_i)-h(t_{i-1})\|.$$

Therefore we have

$$(1-\varepsilon)m(h(t))\,(s(t+r)-s(t))\leqslant(1+\varepsilon)\sum_{i=1}^{n}d(Th(t_i),\,Th(t_{i-1}))=(1+\varepsilon)r.$$

Letting $r\longrightarrow 0^+$, we get

$$m(h(t))\,D^+s(t)\leqslant\frac{1+\varepsilon}{1-\varepsilon}.$$

Since ε is arbitrary, we complete the proof of Lemma 2.

Proof of Theorem 1 *completed.* Let $s_0=\sup\ \{s(t)\,;\ 0\leqslant t<\tau_0\}$. Since s is continuous and strictly increasing, its inverse exists, say $t^*:[0, s_0)\longrightarrow[0, \tau_0)$. Then note that $h(t^*(s))$ is a parametrized curve by arc length with parameter s. Now set

$$F(t)=\int_0^{s(t)}m(h(t^*(s)))\,ds,\quad 0\leqslant t<\tau_0.$$

By Lemma 2, we have $D^+F(t)=m(h(t))\,D^+s(t)\leqslant 1$, thus we obtain, for all t, $0\leqslant t<\tau_0$,

$$\int_0^{s(t)}m(h(t^*(s)))\,ds\leqslant t.$$

Therefore the condition (**) gives that $s_0<\infty$, which also yields that $\lim_{t\to\tau_0}h(t)=x\in\overline{D}$ exists by Remark 1. Now since T has closed graph, $Tx=g(\tau_0)$. If $x\in\partial D$, by assumption (2)

$$\begin{aligned}d(Tx, y)&\geqslant d(Tx_0, y)\\&=d(Tx, y)+\tau_0\\&>d(Tx, y).\end{aligned}$$

This is a contradiction, and it shows that x is in the interior of $\dot{D}$. Therefore by defining $h(\tau_0)=x$, we see that $\tau_0\in M$.

As a direct consequence of Theorem 1, we have the following

COROLLARY 1. *Let X be a Banach space and Y a complete metric space with metric convexity. Let $T: X\longrightarrow Y$ be a locally m-expansive mapping having closed graph, where $m: X\longrightarrow(0, \infty)$ is a continuous function such* *that* $\int_0^\infty m(h(t))dt=\infty$ *for every continuous curve* $h:[0,\infty)\longrightarrow X$ *parametrized by arc length. If* T *maps open subsets of* X *onto open subsets of* Y, *then* $T(X)=Y$.

REMARK 2. Note that if $c:[0,\infty)\longrightarrow(0,\infty)$ is a continuous function for which $\int_0^\infty c(t)dt=\infty$, then the function $m(x)=c(\|x\|)$ satisfies the condition (**). Therefore Theorem 1 contains the results of [3] and [6].

More generally we can give the precise range of the operator T as in [1].

THEOREM 2. *Let* X *be a Banach space,* D *an open subset of* X, *and let* Y *be a complete metric space with metric convexity. Let* x_0 *be in* D, *and* $m:D\longrightarrow(0,\infty)$ *a continuous function such that there is an* $N>0$ *such that*

(***) *if* $h:[0,s)\longrightarrow D$, $0\leqslant s\leqslant\infty$, *is a continuous curve parametrized by arc length for which* $h(0)=x_0$ *and* $\int_0^s m(h(t))dt<N$, *then it follows that* $s<\infty$ *and* $\lim_{t\to s^-} h(t)\in D$.

If $T:D\longrightarrow Y$ *is an open and locally* m*–expansive mapping having closed graph, then* $T(D)$ *contains the ball* $B(Tx_0;N)$.

Proof. Let $y\in B(Tx_0;N)$, that is, $d(y,Tx_0)<N$. As in the proof of Theorem 1, let $g:[0,d(Tx_0,y)]\longrightarrow Y$ be an isometry with $g(0)=Tx_0$ and $g(d(Tx_0,y))=y$, and let M denote the set of all τ in $[0,d(Tx_0,y)]$ for which there exists a unique continuous curve $h:[0,\tau]\longrightarrow D$ such that $h(0)=x_0$ and $Th(t)=g(t)$ for each t, $0\leqslant t\leqslant\tau$. Let $\tau_0=\sup\{\tau;\ \tau\in M\}$. As in the proof of Theorem 1, we get $\tau_0>0$ and we claim that $\tau_0\in M$, so that we conclude that $\tau_0=d(Tx_0,y)$ and $Th(\tau_0)=y\in T(D)$. Also let $h:[0,\tau_0)\longrightarrow D$ be a unique continuous curve with $h(0)=x_0$ and $Th(t)=g(t)$ for each t, $0\leqslant t<\tau_0$. Also by Lemma 1, we let $s(t)$ be the length of the curve $h|_{[0,t]}$ for each $t\in[0,\tau_0)$, and let $s_0=\sup\{s(t);\ 0\leqslant t<\tau_0\}$. Now we know that the inverse $t^*:[0,s_0)\longrightarrow[0,\tau_0)$ of s exists. By applying Lemma 2, we get for each t, $0\leqslant t<\tau_0$,

$$\int_0^{s(t)} m(h(t^*(s)))ds\leqslant t,$$

so that

$$\int_0^{s_0} m(h(t^*(s)))\,ds \leqslant \tau_0 \leqslant d(Tx_0, y) < N.$$

By the condition (***), we have $s_0 < \infty$ and $\lim_{s \to s_0^-} h(t^*(s)) \in D$. Therefore $\lim_{t \to \tau_0^-} h(t) = \lim_{s \to s_0^-} h(t^*(s)) = x \in D$ exists. Thus by defining $h(\tau_0) = x$, we have $\tau_0 \in M$, which completes our proof.

Now if we put $D = B(x_0; K)$, we have the following.

COROLLARY 2. *Let X be a Banach space and Y a complete metric space with metric convexity. Let $x_0 \in X$, $K > 0$, and let $m : B(x_0; K) \subseteq X \longrightarrow (0, \infty)$ be a continuous function such that there is an $N > 0$ such that*

(***)′ *if $h : [0, s) \longrightarrow B(x_0; K)$, $0 \leqslant s \leqslant \infty$, is a continuous curve parametrized by arc length for which $h(0) = x_0$ and $\int_0^s m(h(t))\,dt < N$, then it follows that $s < \infty$ and $\lim_{t \to s^-} \|h(t) - x_0\| < K$.*

If $T : B(x_0; K) \longrightarrow Y$ is an open and locally m-expansive mapping having closed graph, then $T(B(x_0; K))$ contains $B(Tx_0; N)$.

REMARK 3. Note that if $c : [0, K) \longrightarrow (0, \infty)$ is continuous and if $\int_0^K c(t)\,dt = N$, then the function $m(x) = c(\|x - x_0\|)$ satisfies the condition (***)′. Therefore for locally expansive mappings, Theorem 2 is a generalized version of results of [1, 2, 3, 5, 6]. It is interesting to point out that Corollary 1 is also an immediate consequence of Corollary 2. Also note that all results in this paper can be applied to the class of locally strongly ϕ-accretive operators as in [1].

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