J. Korean Math. Soc. **24**(1987), No. 1, pp. 59-63

COMMON FIXED POINTS OF MAPS ON TOPOLOGICAL VECTOR SPACES HAVING SUFFICIENTLY MANY LINEAR FUNCTIONALS

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Fixed point theorems for upper semicontinuous (u. s. c.) multimaps on a nonempty compact convex subset of various topological vector spaces (t. v. s.) were obtained by S. Kakutani [9], Bohnenblust and Karlin [3], Ky Fan [11], Glicksberg [6], and others. Recently, W. K. Kim [10] and S. Park [14] generalized those results for a t. v. s. having sufficiently many linear functionals.

On the other hand, Itoh and Takahashi [7] proved a common fixed point theorem for a continuous map and an u. s. c. multimap on a compact convex subset of a locally convex space (l. c. s.) under some additional conditions.

In the present paper, we generalize their theorem for a t. v. s. having sufficiently many linear functionals, and also obtain a generalized version of the classical Markov-Kakutani theorem.

Let E be a Hausdorff t. v. s. and E^* its topological dual. E is said to have sufficiently many linear functionals if for every $x\in E$ with $x\neq 0$ there exists a continuous linear functional $l\in E^*$ such that $l(x)\neq 0$. By the Hahn-Banach theorem, every l. c. s. has sufficiently many linear functionals. An example of a t. v. s. having sufficiently many linear functionals which is not locally convex is the Hardy space H^p with $0<p<1$.

The following is a consequence of results in [14, 1, 2].

THEOREM 1. *Let K be a nonempty compact convex subset of a Hausdorff t. v. s. E having sufficiently many linear functionals, and $F : K\rightarrow 2^K$ a map such that Fx is nonempty, closed, and convex for each $x\in K$. Then F has a fixed point if one of the following holds:*

Received October 18, 1986.
Supported by a grant from KOSEF, 1986-87.

(i) *F is continuous (u. s. c. and l. s. c.)*
(ii) *E is real and F is u. s. c.*
(iii) *E is locally convex and F is u. s. c.*
(iv) *E is real, locally convex and F is upper hemicontinuous.*

In case where F is a single-valued map $f: K \to K$, Theorem 1 reduces to Ky Fan's theorem in [12]. Note that Theorem 1 includes results of Brouwer [4], Schauder [15], Tychonoff [16], Kakutani [9], Bohnenblust and Karlin [3], Glicksberg [6], Ky Fan [11, 12], and others.

By Theorem 1, the set $A = \mathrm{Fix}(f) = \{x \in K \mid x = fx\}$ is nonempty compact if $f: K \to K$ is continuous, and the set $B = \mathrm{Fix}(F) = \{x \in K \mid x \in Fx\}$ is nonempty compact if one of (i), (ii), and (iii) holds.

We say that f and F commute [7] if for each $x \in K$,

$$f(Fx) \subset F(fx).$$

THEOREM 2. *Under the hypothesis* (i), (ii), *or* (iii) *of* Theorem 1, *if $f: K \to K$ is continuous, f and F commute, and $A = \mathrm{Fix}(f)$ or $B = \mathrm{Fix}(F)$ is convex, then f and F have a common fixed point $z \in K$, that is, $z = fz \in Fz$.*

Proof. Suppose that A is convex. Since $f(Fx) \subset F(fx) = Fx$ for each $x \in A$, f is a continuous selfmap of a nonempty compact convex subset Fx of E. Therefore, by Theorem 1, there is a $y \in Fx$ such that $y = fy$. Hence, $Fx \cap A$ is nonempty. Define a multimap $G: A \to 2^A$ by $Gx = Fx \cap A$ for $x \in A$. If F is continuous [resp. u. s. c.], then G is continuous [resp. u. s. c.] on the nonempty compact convex subset A of E and Gx is nonempty closed convex for each $x \in A$. Thus, by Theorem 1, there exists a fixed point z of G in A. For this z, we have $z = fz \in Fz$.

Suppose that B is convex. For any $x \in B$, we have $fx \in f(Fx) \subset F(fx)$. Hence, f is a continuous selfmap of the nonempty compact convex subset B of E. Therefore, by Theorem 1, there exists a point $z \in B$ such that $z = fz \in Fz$. This completes our proof.

For $f = 1_K$, Theorem 2 reduces to Theorem 1, and for $F = 1_K$, Theorem 2 reduces to Ky Fan's theorem in [12].

A map $F: K \to 2^K$ is said to be affine [7] if for any $x, y \in K$ and $\alpha \in [0, 1]$,

$$\alpha Fx + (1-\alpha) Fy \subset F(\alpha x + (1-\alpha) y).$$

COROLLARY 1. *Under the hypothesis* (i), (ii) *or* (iii) *of* Theorem 1, *if* $f: K \to K$ *is continuous and affine, and* f *and* F *commute, then* f *and* F *have a common fixed point.*

Proof. Since f is affine, the set A is convex.

For $f=1_K$, Corollary 1 reduces to Theorem 1.

COROLLARY 2. *Under the hypothesis* (i), (ii), *or* (iii) *of* Theorem 1, *if* $f: K \to K$ *is continuous,* f *and* F *commute, and* F *is affine, then* f *and* F *have a common fixed point.*

Proof. Since F is affine, the set B is convex.

For $F=1_K$, Corollary 2 redulces to Ky Fan's theorem in [12].

In [7], Itoh and Takahashi proved Theorem 2 and Corollaries 1 and 2 for locally convex E. Our proofs are slight modifications of theirs.

As an application of Theorem 2, we give the Markov-Kakutani theorem for a Hausdorff t. v. s. having sufficiently many linear functionals.

THEOREM 3. *Let* K *be a nonempty compact convex subset of a Hausdorff t. v. s.* E *having sufficiently many linear functionals. Let* $\mathfrak{F}$ *be a commuting family of continuous affine selfmaps of* K. *Then* $\mathfrak{F}$ *has a common fixed point.*

Proof. From Corollary 1, we know that for any $f, g \in \mathfrak{F}$, $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ is nonempty compact convex. Hence so is any finite intersection of sets $\mathrm{Fix}(f)$, $f \in \mathfrak{F}$. Since K is compact, the intersection of all sets $\mathrm{Fix}(f)$ is nonempty.

Theorem 3 for locally convex spaces was first given by Markov [13] with the aid of the Tychonoff fixed point theorem [16]. Kakutani [8] found a direct elementary proof of Theorem 3 (valid in any t. v. s.), and demonstrated the importance of the result by giving a number of applications; he also showed that Theorem 3 implies the Hahn-Banach theorem (see [5]). Our proof of Theorem 3 uses Ky Fan's theorem in [12] (i. e., the single-valued case of Theorem 1).

COROLLARY 3. *Let* $\mathfrak{F}$ *be the same in* Theorem 3 *and* $g: K \to K$ *a continuous map. If* g *commutes with any* $f \in \mathfrak{F}$, *then* $\mathfrak{F} \cup \{g\}$ *has a common fixed point.*

Proof. The set Fix($\mathcal{F}$) of all common fixed points of $\mathcal{F}$ is nonempty compact convex by Theorem 3. Since $gz=gfz=fgz$ for any $z\in$Fix($\mathcal{F}$) and $f\in\mathcal{F}$, g maps Fix($\mathcal{F}$) into itself. Thus g has a fixed point in Fix($\mathcal{F}$) by Theorem 1. This completes our proof.

In fact, Corollary 3 shows that in Theorem 3 we can allow one map $f_0\in\mathcal{F}$ to be non-affine; there will still be a common fixed point for all maps f in $\mathcal{F}$.

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