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RECURRENCE IN PSEUDO-NONEXPANSIVE FLOWS

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1. Introduction

R. Knight [3, 4] proved that a flow on a locally compact Hausdorff space X is recurrent if and only if it is (positively, negatively) Poisson stable and each point is approximated by compact (positive, negative) weak attractors relative to its orbit closure. In this paper, to get another necessary conditions for a Poisson stable flow to be recurrent, we introduce the notions of "(orbitally) Pseudo-nonexpansive flows". It is shown, *in section 2, that the concepts of recurrence and* negatively Poisson stability coincide when the flow is orbitally pseudo-nonexpansive and the phase space X is locally compact. In section 3, a characterization of recurrent motions is obtained under the pseudo-nonexpansive flows, and finally we have some geometric properties of compact minimal sets which are deeply concerned with recurrent orbits in the nonexpansive flows.

Throughout the paper we let (X, π) denote a flow on a metric space X with a metric d. The orbit, orbit closure, limit, and weak attractor relations are denoted, respectively, by $O, \overline{O}, L$, and A_w with the unilateral versions carrying the appropriate $+$ or $-$ superscript. A point x of X is called *recurrent* if and only if, given any $\varepsilon>0$, there is $T \geq 0$ such that $B(x, \varepsilon) \cap y[O, T] \neq \phi$ for every point y in $O(x)$. A point x in X is said to be *positively* (*negatively*) *Poisson stable* provided $x \in L^+(x)$ $(x \in L^-(x))$, and x is (*bilaterally*) *Poisson stable* if it is both positively and negatively Poisson stable. If one of the properties above holds at each point of the phase space X, then the flow (X, π) is said to have that property. A set $M \subset X$ is called *minimal* if and only if it is a closed invariant set containing no nonempty proper subset with these properties.

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2. Orbitally pseudo-nonexpansive flows

DEFINITION 2.1. An orbit $O(x)$, $x \in X$, is called *pseudo-nonexpansive* if and only if there exists a continuous function $\alpha : R \longrightarrow R^+$ with $\alpha(t) \to M$, $0<M<\infty$, as $t \to +\infty$ such that for each $y, z \in O(x)$ and $t \in R^+$, $d(yt, zt) \leq \alpha(t) d(y, z)$. When each orbit is pseudo-nonexpansive, (X, π) is said to be *orbitally pseudo-nonexpansive.*

Throughout the section, α_x denote a continuous function satisfying the assumption of the pseudo-nonexpansive orbit $O(x)$, $x \in X$.

LEMMA 2.2. *The negatively Poisson stable point whose orbit is pseudo-nonexpansive is positively Poisson stable.*

Proof. Let x be a negatively Poisson stable point in X, and let $O(x)$ be pseudo-nonexpansive. Then there is a sequence $\{t_n\}$ in R^- such that $t_n \to -\infty$ and $xt_n \to x$. For such a sequence $\{t_n\}$, it is enough to show that $x(-t_n) \to x$. Suppose $x(-t_n) \not\to x$. Then there is $\varepsilon>0$ such that

$$(1) \qquad d(x, x(-t_n)) > \varepsilon \text{ for all } n.$$

Since $O(x)$ is pseudo-nonexpansive and $xt_n \to x$, there exists n such that

$$d(x, x(-t_n)) \leq \alpha_x(-t_n) d(xt_n, x) < \varepsilon.$$

This contradicts to (1). Consequently x is a positively Poisson stable point.

As we see in Example 2.5, the orbit closure of a negatively poisson stable point need not be minimal. However the orbit closure of a negatively Poisson stable point is minimal if the orbit is pseudo-nonexpansive.

THEOREM 2.3. *The orbit closure of a negatively Poisson stable point whose orbit is pseudo-nonexpansive is minimal.*

Proof. Let x be a negatively Poisson stable point in X, and let $O(x)$ be pseudo-nonexpansive. Suppose $y \in \overline{O(x)}$, Then it is enough to show that $\overline{O(x)} = \overline{O(y)}$. Since $\overline{O(x)}$ is closed and invariant, it is clear that $\overline{O(y)} \subset \overline{O(x)}$. Now we prove that $\overline{O(x)} \subset \overline{O(y)}$: Since x is negatively Poisson stable, we have that $y \in L^-(x)$. Hence there is a sequence $\{t_n\}$

in R^- such that $t_n \to -\infty$ and $xt_n \to y$. Without a loss of generality we may assume that the sequence $\{t_n\}$ is decreasing. Let $z \in \overline{O(x)}$. Then $B(z, \varepsilon) \cap O(x) \neq \phi$ for any $\varepsilon > 0$, and so there is r in R such that $x \in B(z, \varepsilon)(-r)$. Choose $\delta > 0$ such that $\overline{B(x, \delta)} \subset B(z, \varepsilon)(-r)$. Since $O(x)$ is pseudo-nonexpansive and $xt_n \to y$, there is $t_m < 0$ such that if $t_n < t_m$ then

$$d(x,\ x(t_n - t_m)) \leq \alpha_x(-t_m) d(xt_m,\ xt_n) < \delta.$$

Hence we have that $xt_n \in B(x, \delta) t_m$, for each $t_n < t_m$. Consequently we get that

$$y \in \overline{B(x, \delta) t_m} \subset \overline{B(x, \delta)} t_m \subset B(z, \varepsilon)(-r) t_m = B(z, \varepsilon)(t_m - r).$$

Thus we obtain that $B(z, \varepsilon) \cap O(y) \neq \phi$, i.e. $z \in \overline{O(y)}$. This completes the proof.

THEOREM 2.4. *An orbitally pseudo-nonexpansive flow on a locally compact space X is recurrent if and only if it is negatively Poisson stable.*

Proof. Let (X, π) be an orbitally pseudo-nonexpansive and negatively Poisson stable flow, and let x be a point of X. We shall proceed by showing that $\overline{O(x)} = A_w^+(x)$. Let $y \in A_w^+(x)$. Then we have that $x \in L^+(y) = \overline{O(y)} = \overline{O(x)} = L^+(x)$, by Lemma 2.2 and Theorem 2.3. Hence we get that $y \in \overline{O(x)}$. Conversely, let $z \in \overline{O(x)}$. Similarly we obtain that $L^+(x) = \overline{O(x)} = \overline{O(z)} = L^+(z)$, and so $x \in L^+(z)$. Consequently $z \in A_w^+(x)$. Hence we proved that each point of $\overline{O(x)}$ is weakly attracted to x. In [1], it is noted that if each point of $\overline{O(x)}$ is weakly attracted to x then $\overline{O(x)}$ is compact when the phase space X is locally compact. Thus $\overline{O(x)}$ is compact minimal for each $x \in X$. Since each orbit in a compact minimal set is recurrent by [2, Theorem 3.3.8], (X, π) is recurrent. The converse is immediate.

Here we give an example to show that our results make sense.

EXAMPLE 2.5. Let (T, π) be a toral flow given in Example 3.2.7 of [2] consisting of one critical point p and dense regular orbits. In this example, there is exactly one orbit O_1 such that $L^-(O_1) = \{p\}$, and exactly one orbit O_2 such that $L^+(O_2) = \{p\}$. For any other orbit O, $L^+(O) = L^-(O) = T$. Further $L^+(O_1) = L^-(O_2) = T$. Then it is not hard to show that this toral flow (T, π) does not satisfy the properties of the orbitally pseudo-nonexpansive flow in neighborhoods of the critical point p. Morever we know that the points on the orbit O_2 are negatively Poisson stable but not poisitively Poisson stable, and the orbit closure $\overline{O}_2$ is not minimal. Also we can see that this toral flow (T, π) is not recurrent and the only recurrent point is the critical point p. Finally we have that the positive limit set of points of the orbit O_1 is not minimal and also the negative limit set of points of the orbit O_2 is not minimal.

3. Pseudo-nonexpansive flows

DEFINITION 3.1. A flow (X, π) is said to be *pseudo-nonexpansive* provided there exists a continuous function $\alpha : R \longrightarrow R^+$ with $\alpha(t) \to M$, $0<M<\infty$, as $t \to +\infty$ such that for each $x, y \in X$ and $t \in R^+$, $d(xt, yt) \leq \alpha(t) d(x, y)$. Specially, (X, π) is called *nonexpansive* if $\alpha(t) = 1$.

Clearly, the pseudo-nonexpansivity implies the orbitally pseudo-nonexpansivity, but the converse does not hold.

Throughout section, α_π denote a continuous function satisfying the assumption of the pseudo-nonexpansive flow (X, π).

As we know in Example 2.5, the nonempty positive limit sets (or nonempty negative limit sets) of points of X need not be minimal.

THEOREM 3.2. *The nonempty positive limit sets (or nonempty negative limit sets) of points of a pseudo-nonexpansive flow are minimal.*

Proof. Let (X, π) be pseudo-nonexpansive. Suppose $L^+(x) \neq \phi$, $x \in X$, and $y \in L^+(x)$. Then we wish to show that $L^+(x) = \overline{O(y)}$. Since $L^+(x)$ is closed and invariant, it is clear that $\overline{O(y)} \subset L^+(x)$. Suppose $z \in L^+(x) - \overline{O(y)}$. Then there is a sequence $\{t_n\}$ in R^+ such that $t_n \to +\infty$ and $xt_n \to z$. Since $y \in L^+(x)$, there exists a sequence $\{s_n\}$ in R^+ such that $s_n \to +\infty$ and $xs_n \to y$. Since $z \notin \overline{O(y)}$, there is $\varepsilon>0$ such that

$$(1) \qquad d(z, yt) > \varepsilon \text{ for all } t \in R.$$

Since (X, π) is pseudo-nonexpansive, there are t_n and s_n, with $t_n > s_n$, such that

$$d(z, xt_n) < \varepsilon/2 \text{ and } d(y, xs_n) < \varepsilon/2\alpha_\pi(t),$$

where $t=t_n-s_n>0$. Given $t>0$, we have that

$$\begin{aligned} d(z,\ yt) &\leq d(z,\ xt_n)+d(xt_n,\ yt) \\ &\leq d(z,\ xt_n)+d(x(s_n+t),\ yt) \\ &< \varepsilon/2+\alpha_\pi(t)d(xs_n,\ y)<\varepsilon. \end{aligned}$$

This contradicts to (1). Hence we have that $L^+(x)\subset\overline{O(y)}$. Consequently, $L^+(x)$ is minimal. Similarly we can prove that the nonempty negative limit set $L^-(x)$ is minimal.

COROLLARY 3. 3. *Let* (X,π) *be pseudo-nonexpansive and* $L^-(x)\neq\phi$, $x\in X$. *Then the orbit closure* $\overline{O(x)}$ *is minimal.*

Proof. Suppose $y\in L^-(x)$. Then there is a sequence $\{t_n\}$ in R^- such that $t_n\to-\infty$ and $xt_n\to y$. For such a sequence $\{t_n\}$, we can easily see that $y(-t_n)\to x$, using the pseudo-nonexpansivity of $(X,\ \pi)$. Consequently $x\in L^+(y)$. Since $L^-(x)$ is closed and invariant, we have that $x\in L^+(y)\subset\overline{O(y)}\subset L^-(x)$. Hence $L^-(x)=\overline{O(x)}$. By Theorem 3. 2, the orbit closure $\overline{O(x)}$ is minimal.

COROLLARY 3. 4. *Let* (X,π) *be pseudo-nonexpansive. Then a nonempty compact subset* M *of* X *is minimal if and only if* $M=L^+(x)$ *for each* $x\in M$.

Proof. Let $M\subset X$ be minimal, and let $x\in M$. Then $M=\overline{O(x)}$. Since $\overline{O(x)}$ is compact, $L^+(x)\neq\phi$. By minimality of M, $M=L^+(x)$. The converse is trivial by Theorem 3. 2.

THEOREM 3. 5. *A pseudo-nonexpansive flow on a locally compact space* X *is recurrent if and only if each negative limit set of points of* X *is nonempty.*

Proof. Since the necessity is clear, we only show that the sufficiency. Suppose $L^-(x)\neq\phi$, for $x\in X$. Then each orbit closure $\overline{O(x)}$, $x\in X$, is minimal and negatively Poisson stable by Corollary 3. 3. As in the proof of Theorem 2. 4, we can show that $\overline{O(x)}$ is compact. Namely, (X,π) is recurrent.

That the concept of recurrence is deeply concerned with that of compact minimal set is seen from the following Birkhoff's theorem: "Every compact minimal set is the closure of a recurrent orbit" [2, Theorem 3. 3. 8]. Hence the study of compact minimal sets in a flow is crucial in the theory of recurrence. Now we study some properties of compact minimal sets in the nonexpansive flows.

THEOREM 3.6. *Let (X, π) be nonexpansive, and let M and N be two nonempty disjoint compact minimal subsets of X. Then, for any $x \in M$, there exists $y \in N$ satisfying $d(M, N) = d(x, y)$.*

Proof. Choose $a \in M$ and $b \in N$ such that $d(M, N) = d(a, b)$. By Corollary 3.4, we have that $L^+(a) = M$ and $L^+(b) = N$. Let x be an arbitrary point in M. Then there exists a sequence $\{t_n\}$ in R^+ such that $t_n \to +\infty$ and $at_n \to x$. Since $\{bt_n\}$ is a sequence of points in the compact set N, we may assume that the sequence $\{bt_n\}$ converges to a point of N, say $bt_n \to y \in N$. Since (X, π) is nonexpansive, for any given $\varepsilon > 0$, we have that

$$\begin{aligned} d(x, y) &\leqslant d(x, at_n) + d(at_n, bt_n) + d(bt_n, y) \\ &< \varepsilon/2 + d(a, b) + \varepsilon/2 \\ &\leqslant d(a, b) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get that $d(x, y) \leqslant d(a, b)$. This completes the proof.

COROLLARY 3.7. *Let (X, π) be nonexpansive, and let p be a critical point in X. Then any other compact minimal sets in X lie on the surface of a sphere centered at p.*

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