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SINGULARITIES IN THE COMPLEXES OF PSEUDODIFFERENTIAL OPERATORS

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1. Introduction

Treves [7] introduced the complex of pseudodifferential operators of the type

(1) $$D : \Lambda^p C^\infty(\Omega; E) \rightarrow \Lambda^{p+1} C^\infty(\Omega; E).$$

Here Ω is an open subset of R^ν, E is one of the spaces $H^{\pm\infty}$, $E^{0\pm}$ and $D=d_t+d_tB(t, D_x)$ $(t\in\Omega,\ x\in R^n)$. The operator D acting on functions is defined as follows

$$\begin{aligned} Du(t, x) &= d_t u + d_t B(t, D_x) u \\ &= \sum_{j=1}^{\nu} \frac{\partial u}{\partial t_j} dt_j + \sum_{j=1}^{\nu} b_j(t, D_X) u dt_j, \end{aligned}$$

where $d_tB(t, \xi) = \sum_{j=1}^{\nu} b_j(t, \xi) dt_j$ and

$$b_j(t, D_x) u(t, x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} b_j(t, \xi) \hat{u}(t, \xi) d\xi.$$

In particular, when $\nu=n$ and $b_j(t, D_x) = \sqrt{-1}\cdot\frac{\partial}{\partial x_j}$ $(j=1, \cdots, n)$, the operator $\frac{1}{2}D$ becomes the Cauchy-Riemann operator $\bar{\partial}$.

The solvability (at the p-th step in the complex (1)) of the operator D was determined by Treves [7], using the (ψ) condition. The $H^{\pm\infty}$-hypoellipticity of the operator D, in dimension 0, was determined by Maire [6]. In this paper we concern the $H^{\pm\infty}$-hypoellipticity of the operator D. In §3, 4, we obtain the necessary and sufficient condition for the $H^{\pm\infty}$-hypoellipticity (or the $E^{0\pm}$-hypoanalyticity) of D in dimension 0. In §5, we find the necessary and sufficient condition for

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the $H^{\pm\infty}$-hypoellipticity of D in dimension $p(p\geq 1)$.

2. Preliminaries

Let Ω be an open subset of R^{ν} and R_n the dual of an n-dimensional Euclidean space R^n. For any real number s we denote by $H^s=H^s(R^n)$ the standad Sobolev space on R^n, i.e., the space of tempered distributions u in R^n whose Fourier transform $\hat{u}$ is a measurable functions in R_n, satisfying

$$\|u\|_s=(2\pi)^{-n}\left(\int(1+|\xi|^2)^s|\hat{u}(\xi)|^2d\xi\right)^{1/2}<+\infty.$$

Starting with H^s we build the following spaces

$$H^{-\infty}=\bigcup_{s\in R}H^s,\quad H^{+\infty}=\bigcap_{s\in R}H^s.$$

For any real s, let E^s denote the subspace consisting of the generalized functions u whose Fourier transform $\hat{u}$ is a measurable function in R_n satisfying

$$|||\,u\,|||_s=(2\pi)^{-n}\left(\int e^{2s|\xi|}|\hat{u}(\xi)|^2d\xi\right)^{1/2}<+\infty.$$

As with the Sobolev spaces, we form the union and intersection of the spaces E^s, but for s going to zero:

$$E^{0+}=\bigcup_{s>0}E^s,\quad E^{0-}=\bigcap_{s>0}E^{-s}.$$

Let $t=(t_1,\cdots,t_\nu)$ denote the variable point in an open set $\Omega\subset R^{\nu}$. Let E be any one of the spaces $H^{\pm\infty}$, $E^{0\pm}$. If p is any integer such that $0\leq p\leq\nu$, we denote by $\Lambda^pC^\infty(\Omega;E)$ the sapce of C^∞ p-forms valued in the space E. Thus to say that u belongs to $\Lambda^pC^\infty(\Omega;E)$ is the same as the saying that

$$u(t,x)=\sum_{|J|=p}u_J dt_J,$$

where J is an ordered multi-index $(j_1,\cdots,j_p)$ of integers such that $1\leq j_1<\cdots<j_p\leq\nu$, and u_J are C^∞ functions from Ω to E.

Now we consider a C^∞ one form in Ω, depending on the parameter ξ of R_n;

$$b(t,\xi)=\sum_{j=1}^{\nu}b_j(t,\xi)dt_j.$$

We assume that the one form $b(t,\xi)$ is exact in Ω. Thus there exists a primitive B of b such that $b(t,\xi)=d_tB(t,\xi)$. We also assume that

(a) $B(t,\xi)$ is real valued and positive homogeneous of degree one with respect to ξ, and

(b) $B(t,\xi)$ is a C^∞ function of t in Ω with values in $C^1(R_n\backslash 0)$.

We form a pseudodifferential operator

$$D=d_t+b(t,D_x)\Lambda.$$

For each $p(0\leq p\leq \nu-1)$, it defines a linear operator

$$D^p : \Lambda^p C^\infty(\Omega;E) \to \Lambda^{p+1}C^\infty(\Omega;E),$$

and $D^{p+1}\circ D^p=0$ for any $p(0\leq p\leq \nu-1)$. We note that $\hat{D}=e^{-B(t,\xi)}d_t e^{B(t,\xi)}$. It is evident that D, hence also $\hat{D}$, generates a complex.

Our purpose is to study the equations

$$Du=f, \tag{2.1}$$

where f is a $C^\infty(p+1)$-form $(p\leq \nu-1)$ in Ω, or in subsets of Ω, with values in E. Here E is one of the spaces $H^{\pm\infty}$, $E^{0\pm}$. By the Fourier transformation with respect to x we see that (2.1) is equivalent to

$$d_t(e^B\hat{u})=e^B\hat{f} \quad \text{(for a.e. } \xi \text{ in } R_n). \tag{2.2}$$

Thus, if a solution $u\in\Lambda^p\mathcal{D}'(\Omega;FL^2_{\mathrm{loc}})$ (cf. [7]) exists, we must have that

(2.3) for a.e. ξ in R_n, the $(p+1)$-form $e^{B(t,\xi)}\hat{f}(t,\xi)$ is a coboundary.

Vice versa, if (2.3) holds, such a solution u exists, which shows, if it was needed, that (2.3) is independent of the choice of the primitive B. We shall systematically refer to (2.3) as the compatibility conditions for the equation (2.1). We denote by $\mathcal{K}_D{}^{p+1}C^\infty(\Omega;\ E)$ the space of elements of $\Lambda^{p+1}C^\infty(\Omega;\ E)$ which satisfies the compatibility condition (2.3).

3. Singularities of solutions of the homogeneous equations in dimension 0

Throughout this section we shall limit ourselves to the case where the system of operators D acts upon functions, i.e., 0-forms. Thus D is an operator defied by

$$D : C^\infty(\Omega;\ E) \to \Lambda^1 C^\infty(\Omega;E). \tag{3.1}$$

where E one of the four spaces $H^{\pm\infty}$, $E^{0\pm}$.

First we shall consider the null $E^{0\pm}$-hypoanalyticity, the definition of which will be given below. We recall the spaces E^s in the section 2 and their relations:

(3.2) $$E^{0+}=\bigcup_{s>0} E^{s},\quad E^{0-}=\bigcap_{s>0} E^{-s},$$

(3.3) $$E^{k+}=\bigcup_{s>0} E^{k+s},\quad E^{k-}=\bigcap_{s>0} E^{k-s},$$

(3.4) $$\cdots\supset E^{-1}\supset E^{0}\supset E^{1}\supset E^{2}\supset\cdots.$$

DEFINITION 3.1. D is said to be null $E^{0\pm}$-hypoanalytic in Ω if, given any open subset O of Ω

(3.5) $$Du=0,\, u\in C^{\infty}(O;E^{0-})\Rightarrow u\in C^{\infty}(O;\, E^{0+}).$$

THEOREM 3.1. *The following properties are equivalent:*

(3.6) *The operator D is null $E^{0\pm}$-hypoanalytic in Ω.*

(3.7) *Given any open subset O of Ω and $\xi\in R_n\backslash 0$,*
$B(t,\xi)>\inf_{s\in O} B(s,\xi)$ *for all* $t\in O$.

DEFINITION 3.2. The operator D is $H^{\pm\infty}$-hypoelliptic in the open set $\Omega\times P$, $P\subset S_{n-1}$, if D is globally $H^{\pm\infty}$-hypoelliptic in every open subset $O\times P$ of $\Omega\times P$.

DEFINITION 3.3. The operator D is globally null $H^{\pm\infty}$-hypoelliptic in $O\times P\subset\Omega\times S_{n-1}$ if

$$\hat{D}\hat{u}=0 \text{ in } O\times P\Rightarrow u\in C^{\infty}(O\times P).$$

DEFINITION 3.4. The operator D is null $H^{\pm\infty}$-hypoelliptic in the open set $\Omega\times P$, $P\subset S_{n-1}$, if D is globally null $H^{\pm\infty}$-hypoelliptic in every open subset of $\Omega\times P$.

The necessary and sufficient condition for the null $H^{\pm\infty}$-hypoellipticity of D in $O\times P$ will simplify the inequality (1) in Proposition 3 of Maire [6], since $\hat{D}\hat{u}=0$ in $O\times P$.

PROPOSITION 3.1. *The operator D is globally null $H^{\pm\infty}$-hypoelliptic in $O\times P\subset\Omega\times S_{n-1}$ if and only if, to every compact subset $K\times L$ of $O\times P$ there exist a compact set $K'\subset O$ and a constant $C>0$ such that, for any $\rho>0$, $\xi\in L$ and $v\in C^{\infty}(O)$,*

(3.8) $$\rho \sup_{t\in K} e^{-\rho B(t,\xi)}|v(t)|\le C\sup_{t\in K} e^{-\rho B(t,\xi)}|v(t)|.$$

Now we can define that D is null $H^{\pm\infty}$-hypoelliptic in Ω, in dimension 0, if D is globally null $H^{\pm\infty}$-hypoelliptic in any open subset $O\times P$ of $\Omega\times S_{n-1}$. Hence we have the following.

COROLLARY. *The necessary and sufficient condition that D is null $H^{\pm\infty}$-hypoelliptic in Ω, in dimension* 0, *is that D satisfies the inequality* (3.8) *in any open subset $O\times P$ of $\Omega\times S_{n-1}$.*

REMARK. If the inclusion relation

$$(3.9) \qquad E^{0-}\supset H^{-\infty}\supset E^{0}=H^{0}\supset H^{+\infty}\supset E^{0+}$$

holds, then we can show that the necessary and sufficient condition for the null $H^{\pm\infty}$-hypoellipticity of D in Ω in dimension 0, is equivalent to the condition (3.7). Unfortunately the relation (3.9) does not hold and hence the condition (3.7) can not guarantee the null $H^{\pm\infty}$-hypoellipticity of D in Ω, in dimension 0.

4. Singularities of solutions of the inhomogeneous equations in dimension 0

In this section we look at the singularities of the solution $u\in C^{\infty}(\Omega;E)$ of the equations, in Ω,

$$(4.1) \qquad Du=f,$$

where f is a given element of $\Lambda^1C^{\infty}(\Omega;E)$. Here E is one of the spaces $H^{-\infty}$, E^{0-}. When we refer below to the operator D, we regard it as an operator

$$D:\Lambda^0C^{\infty}(\Omega;E)\rightarrow\Lambda^1C^{\infty}(\Omega;E).$$

Here E is one of the spaces $H^{\pm\infty}$, $E^{0\pm}$. The sets $SS(u)$ and $SA(u)$ are defined at the end of the section 1.1 in Choi [1].

DEFINITION 4.1. We say that the operator D is $E^{0\pm}$-hypoanalytic in Ω, in dimension 0, if given any open subset O of Ω and any function $u\in C^{\infty}(O;E^{0-})$,

$$(4.2) \qquad Du\in\Lambda^1C^{\infty}(O;E^{0+})\Rightarrow u\in C^{\infty}(O;E^{0+}).$$

THEOREM 4.1. *The following conditions are the nccessary and sufficient conditions for D to be $E^{0\pm}$-hypoanalytic in Ω, in dimension* 0.

(4.3) *Given any open subset O of Ω and any $f\in Im\ D\cap \mathscr{B}_D{}^1(O;E^{0+})$, there exists at least on $U\in C^{\infty}(O;E^{0+})$ with $DU=f$.*

(4.4) *Given any open subset O of Ω and any $\xi\in S_{n-1}$, $B(t,\xi)>\inf_{s\in O}B(s,\xi)$ holds for all $t\in O$.*

Proof. Suppose that the conditions (4.3), (4.4) hold. Assume that given any open subset O of Ω, $u\in C^\infty(O;E^{0-})$ and $Du=f\in \Lambda^1C^\infty(O;E^{0+})$. Then $Du\in \mathrm{Im}D\cap \mathscr{B}_D{}^1C^\infty(O;E^{0+})$. By the assumption (4.3), there exists $u_1\in C^\infty(O;E^{0+})$ with $Du_1=Du$. On the other hand u_1-u is a solution of $Du=0$ and hence by the assumption (4.4) $u_1-u\in C^\infty(O;E^{0+})$. Therefore $u=u_1+(u-u_1)$ belongs to $C^\infty(O;E^{0+})$.

Conversely if (4.3) or (4.4) does not hold, then clearly D is not $E^{0\pm}$-hypoanalytic.

DEFINITION 4.2. We say that the operator D is $H^{\pm\infty}$-hypoelliptic in Ω, in dimension 0, if given any open subset O of Ω and any function $u\in C^\infty(O;H^{-\infty})$,

(4.5) $$Du\in \Lambda^1C^\infty(O;H^{+\infty})\Rightarrow u\in C^\infty(O;H^{+\infty}).$$

THEOREM 4.2. *The following conditions are the ncessary and sufficient conditions for D to be $H^{\pm\infty}$-hypoelliptic in Ω, in dimension* 0.

(4.6) *Given any open subset O of Ω, for any $f\in Im\ D\cap \mathscr{B}_D{}^1C^\infty(O;H^{+\infty})$, there exists at least one $U\in C^\infty(O;H^{+\infty})$ with $DU=f$.*

(4.7) *Given any open subset $O\times P$ of $\Omega\times S_{n-1}$, to every compact subset $K\times L$ of $O\times P$ there exist a compact $K'\subset O$ and a constant $C>0$ such that, for any $\rho>0$, $\xi\in L$ and $v\in C^\infty(O)$,*

$$\rho \sup_{t\in K} e^{-\rho B(t,\xi)}|v(t)|\leq C\sup_{t\in K'} e^{-\rho B(t,\xi)}|v(t)|.$$

The proof of this theorem is similar to that of Theorem 4.1.

REMARK. Under the hypothesis that D has property $\psi(0)$ stated in Kim [5] the necessary and sufficient condition for the $E^{0\pm}$-hypoanalyticity (or $H^{\pm\infty}$-hypoellipticity) of D in Ω, in dimension 0, is equivalent to the condition (4.4) (or (4.7)).

EXAMPLE 4.1. Let $B(t,\xi)=(t_1{}^3-t_2{}^3)\xi$, $t\in R^2$, $\xi\in R_n$. Then, clearly, $B(t,\xi)$ satisfies the condition (4.4) and (4.7) for all $\xi\in S_{n-1}$, in an open set $\Omega=(-T_1,T_1)\times(-T_2,T_2)\subset R^2$. Given $Du=f$ with $f\in \mathscr{B}_D{}^1C^\infty(\Omega;E)$, we can construct a solution $u\in C^\infty(\Omega;E)$ of this equation using the integral representation in Choi [1]. Here E is one of the spaces $H^{+\infty}$, E^{0+}. Therefore D satisfies the conditions (4.3), (4.6). So D is $E^{0\pm}$-hypoanalytic and $H^{\pm\infty}$-hypoelliptic in Ω, in dimension 0.

EXAMPLE 4.2. The simple operator $D=\partial/\partial t-it^2\,\partial/\partial x$ in $(-T,T)$,

$x\in R^1$, is $E^{0\pm}$-hypoanalytic and $H^{\pm\infty}$-hypoelliptic since $B(t,\xi)=t^3\xi/3$ satisfies the conditions (4.3), (4.4), (4.6), (4.7). That this operator D is hypoanalytic and hypoelliptic in the usual sense in a well known fact.

5. Singularities in dimension $p(p\geq 1)$

We consider the complex of sheaves

$$(5.1)\qquad \cdots\to \Lambda^p C^\infty(\Omega;E)\xrightarrow{D^p}\Lambda^{p+1}C^\infty(\Omega;E)\to\cdots,$$

$p=0,1,\cdots,\nu-1$. Here E is one of the spaces $H^{\pm\infty}$, $E^{0\pm}$. When $p>0$, we note that given any $u\in\Lambda^pC^\infty(\Omega;H^{-\infty})$ and any $f\in\Lambda^{p+1}C^\infty(\Omega;H^{-\infty})$, the equation

$$Du=f,\ f\in\mathscr{B}_D{}^{p+1}C^\infty(\Omega;H^{+\infty}),$$

can not guarantee $u\in\Lambda^pC^\infty(\Omega;H^{+\infty})$ even if $B(t,\xi)$ is sufficiently nice. In fact, if $u\in\mathscr{B}_D{}^pC^\infty(\Omega;H^{-\infty})$ and $p>0$, then $Du=0$. So there can be an element $v\in\Lambda^pC^\infty(\Omega;H^{-\infty})$, $p>0$, such that $Dv=0$ but $v\notin\Lambda^pC^\infty(\Omega;H^{+\infty})$, even if $B(t,\xi)$ is a nice function. Therefore the natural generalization of the $H^{\pm\infty}$-hypoellipticity in Ω, in dimension $p(p>0)$, can be stated as follows; namely, if, given any open subset O of Ω,

$$Du=f \text{ in } O \text{ and } f\in\Lambda^{p+1}C^\infty(O;H^{+\infty}),$$

then there exists $v\in\Lambda^{p-1}C^\infty(O;H^{-\infty})$ such that

$$u-Dv\in\Lambda^pC^\infty(O;H^{+\infty}).$$

The above property can be stated in microlocal version. We note that the space of micro p-forms in O is by definition

$$\Lambda^pC^\infty(O;H^{-\infty})/\Lambda^pC^\infty(O;H^{+\infty})$$

Then our complex $\{\Lambda^pC^\infty(O;H^{-\infty});D\}$ induces in a natural way, a complex of micro forms, namely,

$$(5.2)\qquad \dot{D}^p:\Lambda^pC^\infty(O;H^{-\infty})/\Lambda^{p+1}C^\infty(O;\mathrm{H}^{+\infty})\to\Lambda^{p+1}C^\infty(O;H^{-\infty})/\Lambda^{p+1}C^\infty(O;H^{+\infty}).$$

Via the system $\dot{D}^p$ we can define $H^{\pm\infty}$-hypoellipticity in p-th step$(p>0)$ as follows.

DEFINITION 5.1. D is $H^{\pm\infty}$-hypoelliptic in dimension p, $p=1,2,\cdots,\nu-1$, in Ω if, given any open subset O of Ω

$$\mathrm{Im}\,\dot{D}^{p-1}=\mathrm{Ker}\,\dot{D}^p$$

in the complex (5.2).

These ideas for the definition of the $H^{\pm\infty}$-hypoellipticity in dimension $p (p>0)$ are quoted from the paper of Kim [3].

We note that the above definition is a true generalization of the $H^{\pm\infty}$-hypoellipticity defined when $p=0$ in Definition 5.2 if we set $\Lambda^{-1}C^{\infty}(O; E)=0$ for $E=H^{+\infty}$ or $H^{-\infty}$.

We consider the following three complexes

$$(5.3)\quad 0 \longrightarrow \operatorname{Ker} D^0 \xrightarrow{i} C^{\infty}(O; H^{-\infty}) \xrightarrow{D^0} \Lambda^1 C^{\infty}(O; H^{-\infty}) \xrightarrow{D^1} \cdots,$$

$$(5.4)\quad 0 \longrightarrow \operatorname{Ker} D^0|_{H^{+\infty}} \xrightarrow{i} C^{\infty}(O; H^{+\infty}) \xrightarrow{D^0|_{H^{+\infty}}} \Lambda^1 C^{\infty}(O; H^{+\infty}) \xrightarrow{D^1|_{H^{+\infty}}} \cdots,$$

$$(5.5)\quad 0 \longrightarrow \operatorname{Ker} D^0/\operatorname{Ker} D^0|_{H^{+\infty}} \xrightarrow{i} C^{\infty}(O; H^{-\infty})/C(O; H^{+\infty}) \xrightarrow{\dot{D}^0} \Lambda^1 C^{\infty}(O; H^{-\infty})/\Lambda^1 C^{\infty}(O; H^{+\infty}) \xrightarrow{\dot{D}^1} \cdots,$$

where $O, i, D^p|_{H^{+\infty}}$ denote any open subset of Ω, an inclusion, a restriction of D^p to $\Lambda^p C^{\infty}(O; H^{+\infty})$, respectively. Let $\mathcal{A}=a$ cochain complex (5.3),

$\mathcal{B}$=a cochain complex (5.4),

$\mathcal{C}$=a cochain complex (5.5).

In the following commutative diagram the row sequences are exact

(5.6)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker} D^0_{H^{+\infty}} & \xrightarrow{i} & \operatorname{Ker} D^0 & \xrightarrow{r} & \operatorname{Ker} \dot{D}^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^{\infty}(O; H^{+\infty}) & \xrightarrow{i} & C^{\infty}(O; H^{-\infty}) & \xrightarrow{r} & C^{\infty}(O; H^{-\infty})/C^{\infty}(O; H^{+\infty}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda^1 C^{\infty}(O; H^{+\infty}) & \xrightarrow{i} & \Lambda^1 C^{\infty}(O; H^{-\infty}) & \xrightarrow{r} & \Lambda^1 C^{\infty}(O; H^{-\infty})/\Lambda^1 C^{\infty}(O; H^{+\infty}) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

That is, $0 \to \mathcal{B} \to \mathcal{A} \to \mathcal{C} \to 0$ is an exact sequence.

Therefore we obtain a long exact cohomology sequence

(5.7)

$$0 \longrightarrow H^0(\mathcal{B}) \xrightarrow{i_{0*}} H^0(\mathcal{A}) \longrightarrow H^0(\mathcal{C}) \xrightarrow{\partial_0} H^1(\mathcal{B}) \longrightarrow \cdots \longrightarrow H^{p-1}(\mathcal{C}) \xrightarrow{\partial_{p-1}} H^p(\mathcal{B}) \xrightarrow{i_{p*}} H^p(\mathcal{A}) \xrightarrow{r_p} H^p(\mathcal{C}) \xrightarrow{\partial_p} H^{p+1}(\mathcal{B}) \longrightarrow \cdots,$$

where ∂_p is a connecting homomorphism. Here i_p*, r_p are induced homomorphisms. The cohomology space in the sequence (5.7) are defined in the standard manner; that is,

$H^0(\mathscr{B})=\text{Ker } D^0|_{H^{+\infty}}$, $H^0(\mathscr{A})=\text{Ker } D^0$, $H^0(\mathscr{C})=\text{Ker } \dot{D}^0$, if $p>0$,
$H^p(\mathscr{B})=\text{Ker } D^p|_{H^{+\infty}}/\text{Im } D^{p-1}|_{H^{+\infty}}$,
$H^p(\mathscr{A})=\text{Ker } D^p/\text{Im } D^{p-1}$,
$H^p(\mathscr{C})=\text{Ker } \dot{D}^p/\text{Im } \dot{D}^{p-1}$.

REMARK 5.1. Definition 5.1 that D is $H^{\pm\infty}$-hypoelliptic in dimension p, $p=1,\cdots,\nu-1$, in Ω is equivalent to $H^p(\mathscr{C})=0$. In paticular the $H^{\pm\infty}$-hypoellipticity of D^0 is equivalent to $H^0(\mathscr{C})=0$.

First we consider the $H^{\pm\infty}$-hypoellipticity of D^0. By the following exact sequence

$$0 \longrightarrow H^0(\mathscr{B}) \xrightarrow{i_{0*}} H^0(\mathscr{A}) \longrightarrow H^0(\mathscr{C}) \longrightarrow H^1(\mathscr{B}) \xrightarrow{i_{1*}} H^1(\mathscr{A}) \longrightarrow \cdots$$

the necessary and sufficient condition for $H^0(\mathscr{C})=0$ is that $H^0(\mathscr{A})=H^0(\mathscr{B})$ and $i_{1*}: H^1(\mathscr{B}) \to H^1(\mathscr{A})$ a monomorphism. We recall that $H^0(\mathscr{A})=\text{Ker } D^0$, $H^0(\mathscr{B})=\text{Ker } D^0$, $H^0(\mathscr{B})=\text{Ker } D^0|_{H^{+\infty}}$. We note that

$H^0(\mathscr{A})=H^0(\mathscr{B})$ for any given open subset O of Ω
$\Longleftrightarrow$ Ker $D^0=$ Ker $D^0|_{H^{+\infty}}$ for any given open subset O of Ω
$\Longleftrightarrow$ $D^0u=0$, $u\in C^\infty(O; H^{-\infty})$ implies that
$u\in C^\infty(O; H^{+\infty})$ for any given open subset of Ω
$\Longleftrightarrow$ D^0 is null $H^{\pm\infty}$-hypoelliptic in Ω.

Therefore the necessary and sufficient condition for $H^0(\mathscr{A})=H^0(\mathscr{B})$ is equivalent to the necessary and sufficient condition for the null $H^{+\infty}$-hypoellipticity of D^0. Thus, by Proposition 3.1, the necesay and sufficient condition for $H^0(\mathscr{A})=H^0(\mathscr{B})$ is that for any $\xi\in S_{n-1}$ and all $t\in O$, $B(t,\xi)$ satisfies the inequality (3.7).

On the other hand, that the map $i_{1*}: H^1(\mathscr{B}) \to H^1(\mathscr{A})$ is a monomorphism is equivalent to that for any $f\in \text{Im } D^0\cap \mathscr{B}_D{}^1C^\infty(O; H^{+\infty})$ there exists at least one $U\in C^\infty(O; H^{+\infty})$ with $DU=f$. Therefore we have

THEOREM 5.1. *D^0 is $H^{\pm\infty}$-hypoelliptic in Ω if and only if the following conditions (a) and (b) hold.*

(a) *Given any open subset O of Ω, for any $f\in ImD^0\cap \mathscr{B}_D{}^1C^\infty(O; H^{+\infty})$ there exists at least one $U\in C^\infty(O;H^{+\infty})$ with $D^0U=f$.*

(b) *Given any open subset $O\times P$ of $\Omega\times S_{n-1}$, to every compact subset $K\times L$ of $O\times P$ there exist a compact $K'\subset O$ and a constant $C>0$*

such that, for any $\rho > 0$, $\xi \in L$ *and* $v \in C^\infty(O)$

$$\rho \sup_{t\in K} e^{-B(t,\xi)} |v(t)| \leq C \sup_{t\in K'} e^{-\rho B(t,\xi)} |v(t)|.$$

REMARK 5.2. Theorem 5.1 is the same as Theorem 4.2.

Next we consider the $H^{\pm\infty}$-hypoellipticity of $D^p (p>0)$ in Ω. We recall that the necessary and sufficient condition for the $H^{\pm\infty}$-hypoellipticity of D^p in Ω is that $H^p(\mathscr{C})=0$ for any given open subset O of Ω.

We consider the following exact sequence

$$\text{(5.8)} \quad \cdots \longrightarrow H^{p-1}(\mathscr{C}) \xrightarrow{\partial} H^p(\mathscr{B}) \xrightarrow{i_{p*}} H^p(\mathscr{A}) \longrightarrow H^p(\mathscr{C}) \longrightarrow H^{p+1}(\mathscr{B}) \xrightarrow{i_{p+1*}} H^{p+1}(\mathscr{A}) \longrightarrow \cdots.$$

The necessary and sufficient condition that $H^p(C)=0$ is that $i_p* : H^p(\mathscr{B}) \to H^p(\mathscr{A})$ is an epimorphism and $i_{p+1}* : H^{p+1}(\mathscr{B}) \to H^{p+1}(\mathscr{A})$ is a monomorphism. Therefore we obtain the following theorem.

THEOREM 5.2. *D is $H^{\pm\infty}$-hypoelliptic, in dimension $p(p>0)$, in Ω if and only if given any open subset O of Ω, in the sequence* (5.8).

$i_p* : H^p(\mathscr{B}) \to H^p(\mathscr{A})$ is an epimorphism and
$i_{p+1}* : H^{p+1}(\mathscr{B}) \to H^{p+1}(\mathscr{A})$ is a monomorphism.

REMARK 5.3. In the above we set

$H^p(\mathscr{B}) = \text{Ker } D^p|_{H^{+\infty}}/\text{Im } D^{p-1}|_{H^{+\infty}}$ and
$H^p(\mathscr{A}) = \text{Ker } D^p/\text{Im } D^{p-1}$.

That $i_p* : H^p(\mathscr{B}) \to H^p(\mathscr{A})$ is an epimorphism is equivalent to the fact that for any $f \in \text{Ker } D^p$ there exists at least one $f' \in \text{Ker } D^p|_{H^{+\infty}}$ with $f-f' \in \text{Im } D^{p-1}$.

On the other hand, that $i_{p+1}* : H^{p+1}(\mathscr{B}) \to H^{p+1}(\mathscr{A})$ is a monomorphism is equivalent to the fact that given any open subset O of Ω, for any $f \in \text{Im } D^p \cap \mathscr{B}_D{}^{p+1}C^\infty(O; H^{+\infty})$ there exists at least one $U \in \Lambda^p C^\infty(O; H^{+\infty})$ with $D^p u = f$. Therefore Theorem 5.2 can be restated as follows.

THEOREM 5.2$'$. *D is $H^{\pm\infty}$-hypoelliptic, in dimension $p(p>0)$, in Ω if and only if the following conditions (a) and (b) hold.*

(*a*) *Given any open subset O of Ω, for any $f \in Ker\ D^p$ there exists at least one $f' \in Ker\ D^p|_{H^{+\infty}}$ with $f-f' \in Im\ D^{p-1}$*

(*b*) *Given any open subset O of Ω, for any $f \in Im\ D^p \cap \mathscr{B}_D{}^{p+1}C^\infty(O; H^{+\infty})$ there exists at least one*

$U \in \Lambda^p C^\infty(O; H^{+\infty})$ *with* $D^p U = f$.

EXAMPLE. Let $B(t, \xi) = (t_1^3 - t_2^3)\xi$ $(\xi \in R_n)$ and $D = d_t + d_t\ B(t, D_x)$. Then D is $H^{\pm\infty}$-hypoelliptic, in dimension 1, in $\Omega = (-T_1, T_1) \times (-T_2, T_2) \subset R^2$.

Proof. Given any open subset O of Ω, let f be an element of Ker D^1 for the given open set O. Take any element $f' \in$ Ker $D^1|_{H^{+\infty}}$ for the open set O. To show that D satisfies the condition (a) in Theorem 5.2′, in dimension 1, it suffices to show that for any $t_0 \in O$ there exist an open neighborhood O' of t_0 in O and $U \in C^\infty(O'; H^{-\infty})$ with $D^0 U = f - f'$ in O'. But we can construct such U using the integral representation in Choi [1]. Therefore D satisfies the condition (a) in Theorem 5.2′, in dimension 1. Also we can see that D satisfies the condition (b) in Theorem 5.2′, in dimension 1, by using the integral representation in Choi [1]. Hence D is $H^{\pm\infty}$-hypoelliptic, in dimension 1, in Ω.

REMARK 5.4. If we follow all the discussions in this section with replating $H^{\pm\infty}$ by $E^{0\pm}$, we can obtain the necessary and sufficient condition for the $E^{0\pm}$-hypoanalyticity of D^p $(p = 1, 2, \cdots, \nu - 1)$ in Ω with replacing $H^{\pm\infty}$ by $E^{0\pm}$ in Theorem 5.2 and Theorem 5.2′.

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