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SPECTRA OF WEIGHT SHIFT OPERATORS ON THE SPACE $l_2(q)$

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1. Introduction

In this paper, we proved mainly that the spectra of weight shift operators on the space $l_2(q)$ depend on the weight and q, where $l_2(q)$ is the set of all sequences $x=(x_1, x_2, x_3, \cdots\cdots)$ such that $x_n \in C$ for all n and $\sum_{n=1}^{\infty}|x_n|^2 q^{-n}<\infty$. Throughout this paper, we denote $L(E)$ for the set of all bounded linear operators, where E is a normed space over the complex field $\boldsymbol{C}$, we write $\sigma(A)$, $\sigma_{\mathrm{p}}(A)$, $\sigma_{\mathrm{com}}(A)$, $\sigma_{\mathrm{ap}}(A)$, $\sigma_{\mathrm{r}}(A)$ and $\sigma_{\mathrm{c}}(A)$ for the spectrum, point spectrum, compression spectrum, approximate point spectrum, residual spectrum and continuous spectrum of an operator $A \in L(E)$ respectively.

It is well-known that $\sigma_{\mathrm{r}}(A)=\sigma_{\mathrm{com}}(A)\backslash\sigma_{\mathrm{p}}(A)$, $\sigma_{\mathrm{c}}(A)=\sigma(A)\backslash\{\sigma_{\mathrm{p}}(A)\cup\sigma_{\mathrm{com}}(A)\}$ and $\sigma(A)=\sigma_{\mathrm{p}}(A)\cup\sigma_{\mathrm{r}}(A)\cup\sigma_{\mathrm{c}}(A)$, where $\sigma_{\mathrm{p}}(A)$, $\sigma_{\mathrm{r}}(A)$ and $\sigma_{\mathrm{c}}(A)$ are mutually disjoint.

2. Main results

We will investigate the spactra of the shift operators on the Banach space $l_1(q)$ and the spectra of the weight shift operators on the Hilbert space $l_2(q)$.

We begin with the first part. Let $q>0$ be given, and let $l_1(q)$ be the set of all sequences $x=(x_1, x_2, x_3, \cdots\cdots)$ such that $\sum_{n=1}^{\infty}|x_n|q^{-n}<\infty$ and $x_n \in \boldsymbol{C}$ for all n. we define a norm $\|x\|_q=\sum_{n=1}^{\infty}|x_n|q^{-n}$, then clearly $l_1(q)$ is a Banach space.

THEOREM 2.1. *Let A be the left shift operator on $l_1(q)$. Then we have the following;*

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(1) $\sigma(A)=\sigma_{ap}(A)=\{\lambda\in \boldsymbol{C}: |\lambda|\leqq q\}$ and $\sigma_p(A)=\{\lambda\in \boldsymbol{C}: |\lambda|<q\}$,
(2) $\sigma_{com}(A)=\sigma_r(A)=\phi$ *and* $\sigma_c(A)=\{\lambda\in \boldsymbol{C}: |\lambda|=q\}$.

Proof. Clearly A is a bounded linear operator with $\|A\|=q$. Suppose that $Ax=\lambda x$ for $x=(x_n)\in l_1(q)$. Then $(x_2, x_3, x_4, \cdots\cdots)=(\lambda x_1, \lambda x_2, \lambda x_3, \cdots\cdots)$, that is; $x_{n+1}=\lambda x_n$ for all n. If $x_1=0$, then $x=0$ in $l_1(q)$. Let $\lambda\in\sigma_p(A)$. Then there exists a non-zero element $x=(x_n)$ in $l_1(q)$ such that $Ax=\lambda x$. Since $x_1\neq 0$, $\|\lambda x\|_q=\|Ax\|_q<q\|x\|_q$ and so $\sigma_p(A)\subset\{\lambda\in \boldsymbol{C}: |\lambda|<q\}$. Let $|\lambda|<q$, and let $x=(q^{-1}, \lambda q^{-1}, \lambda^2 q^{-1}, \cdots\cdots)$. Then $\|x\|_q= q^2 (\sum_{n=1}^{\infty}\left|\frac{\lambda}{q}\right|^n)<\infty$, $Ax=\lambda x$ and so $\{\lambda\in \boldsymbol{C}: |\lambda|<q\}\subset\sigma_p(A)$. Hence $\sigma_p(A)=\{\lambda\in \boldsymbol{C}: |\lambda|<q\}$.

It is well-known that $\sigma_p(A)\subset\sigma_{ap}(A)\subset\sigma(A)\subset\{\lambda\in \boldsymbol{C}: |\lambda|\leqq\|A\|=q\}$. Since $\sigma_{ap}(A)$ and $\sigma(A)$ are closed subsets of $\boldsymbol{C}$, we have $\sigma_{ap}(A)=\sigma(A)=\{\lambda\in \boldsymbol{C}: |\lambda|\leqq q\}$.

For (2), it is known that $\sigma_{com}(A)=\sigma_p(A')$, where A' is the conjuugate operator of A. Let $\lambda\in\sigma_p(A')$. Then there exists a non-zoro element f in the dual space $[l_1(q)]'$ of $l_1(q)$ such that $A'f=\lambda f$. Since $Ae_1=0$, $Ae_{n+1}=qe_n$ for all n and $f(Ae_n)=\lambda f(e_n)$ where $\{e_n : e_n=(0,0,0,\cdots, q^n, 0,0,0,\cdots), n=1,2,3,\cdots\cdots\}$. Thus $f=0$, a contradiction. It follows that $\sigma_{com}(A)=\sigma_p(A')=\phi$. Moreover, $\sigma_r(A)=\sigma_{com}(A)\backslash\sigma_p(A)=\phi$, $\sigma_c(A)=\sigma(A)\backslash\{\sigma_{com}(A)\cup\sigma_p(A)\}=\{\lambda\in \boldsymbol{C}: |\lambda|=q\}$.

THEOREM 2.2. *Let V be the right shift operator on $l_1(q)$. Then we have the following;*

(1) $\sigma_p(V)=\phi$, $\sigma_{com}(V)=\left\{\lambda\in \boldsymbol{C}: |\lambda|<\frac{1}{q}\right\}$ *and*

$$\sigma(V)=\left\{\lambda\in \boldsymbol{C}: |\lambda|\leqq\frac{1}{q}\right\},$$

(2) $\sigma_{ap}(V)=\sigma_c(V)=\left\{\lambda\in \boldsymbol{C}: |\lambda|=\frac{1}{q}\right\}$ *and* $\sigma_r(V)=\sigma_{com}(V)$.

From (1) and (2), we see that $\sigma_{ap}(V)\cap\sigma_{com}(V)=\phi$.

Proof. Clearly V is a bounded linear operator with $\|V\|=\frac{1}{q}$. *If* $\lambda\in\sigma_p(V)$, then there exists a non-zoro element $x=(x_n)$ in $l_1(q)$ such that $Vx=\lambda x$. Thus $(0, x_1, x_2, x_3, \cdots\cdots)=(\lambda x_1, \lambda x_2, \cdots\cdots)$, that is; $\lambda x_1=0$ and $\lambda x_{n+1}=x_n$ for all n. This implies $x=0$ in $l_1(q)$, a contradiction. Hence $\sigma_p(V)=\phi$. It is known that $\sigma_{com}(V)=\sigma_p(V')$ where V' is the conjugate operator of V. Let $|\lambda|<\frac{1}{q}$. We define $f: l_1(q)\to \boldsymbol{C}$ by $f(x) =\sum_{n=1}^{\infty} x_n\lambda^{n-1}$ for $x=(x_n)$, then $|f(x)|\leqq\sum_{n=1}^{\infty}|x_n||\lambda|^{n-1}\leqq\sum_{n=1}^{\infty}|x_n|q^{-n+1}=q\|x\|_q <\infty$. Thus f is a bounded linear functional on $l_1(q)$ and $V'f=\lambda f$. Hence $\left\{\lambda\in \boldsymbol{C}: |\lambda|<\frac{1}{q}\right\}\subset\sigma_p(V')$. For a $\lambda\in\sigma_p(V')$ there exists a non-zero bounded linear functional f on $l_1(q)$ such that $V'f=\lambda f$, and so $|\lambda|<\frac{1}{q}$. Therefore $\sigma_{com}(V)=\{\lambda\in \boldsymbol{C}: |\lambda|<\frac{1}{q}\}$. Since $\sigma_{com}(V)\subset\sigma(V) \subset\left\{\lambda\in \boldsymbol{C}: |\lambda|\leqq\|V\|=\frac{1}{q}\right\}$, $\sigma(V)=\left\{\lambda\in \boldsymbol{C}: |\lambda|\leqq\frac{1}{q}\right\}$.

(2) It is known that $\left\{\lambda\in \boldsymbol{C}: |\lambda|=\frac{1}{q}\right\}=\partial\sigma(V)\subset\sigma_{ap}(V)$. Let $|\lambda|\neq \frac{1}{q}$. Then $\|(V-\lambda I)x\|_q\geqq|\ \|Vx\|_q-\|\lambda x\|_q\ |=|\frac{1}{q}-|\lambda|\ |\|x\|_q$ for all $x\in l_1(q)$. Thus $V-\lambda I$ is bounded below, and so $\lambda\notin\sigma_{ap}(V)$. Hence $\sigma_{ap}(V)=\left\{\lambda\in \boldsymbol{C}: |\lambda|=\frac{1}{q}\right\}$. Moreover, $\sigma_r(V)=\sigma_{com}(V)\backslash\sigma_p(V)=\sigma_{com}(V)$. $\sigma_c(V)=\sigma(V)\backslash\{\sigma_{com}(V)\cup\sigma_p(V)\}=\left\{\lambda\in \boldsymbol{C}: |\lambda|=\frac{1}{q}\right\}$.

Now, we discuss the spectra of the weight shift operators on the Hilbert space $l_2(q)$.

PROPOSITION 2.3. *Let $q>0$ be given, and let $l_2(q)$ be the set of all sequences $x=(x_1, x_2, x_3, \cdots\cdots)$ such that $x_n\in \boldsymbol{C}$ for all n and $\sum_{n=1}^{\infty}|x_n|^2q^{-n} <\infty$. Define an inner product of vectors $x=(x_n)$ and $y=(y_n)$ by $(x|y) =\sum_{n=1}^{\infty}x_n\bar{y}_nq^{-n}$. Then $l_2(q)$ becomes a Hilbert space.*

Proof. Obviously $\|x\|_q=0$ if and only if $x=0$ in $l_2(q)$, $\|\alpha x\|_q=|\alpha| \|x\|_q$ for all $\alpha\in \boldsymbol{C}$ and $x\in l_2(q)$. For the triangular inequality, let $x=(x_n)$ and $y=(y_n)$ be elements of $l_2(q)$. By the Cauchy-Schwarz inequality, $\sum_{n=1}^{\infty}|x_ny_n|q^{-n}\leqq\left(\sum_{n=1}^{\infty}|x_n|^2q^{-n}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}|y_n|^2q^{-n}\right)^{\frac{1}{2}}$ we have $\|x+y\|_q\leqq \left\{\sum_{n=1}^{\infty}|x_n|^2q^{-n}+2\sum_{n=1}^{\infty}|x_ny_n|q^{-n}+\sum_{n=1}^{\infty}|y_n|^2q^{-n}\right\}^{\frac{1}{2}}\leqq\|x\|_q+\|y\|_q$.

Let (x_n) be any Canchy sequence in the space $l_2(q)$, where $x_n=(\alpha_1^{(n)}, \alpha_2^{(n)}, \cdots)$. Then for any $\varepsilon>0$, there exists $N\in \boldsymbol{N}$ such that $\left(\sum_{j=1}^{\infty}|\alpha_j^{(n)} -\alpha_j^{(m)}|^2q^{-j}\right)^{\frac{1}{2}}<\varepsilon$ for $n, m>N$. It follows that

(*) $|(\alpha_j^{(n)}-\alpha_j^{(m)})q^{-j}|<\varepsilon$ for every $j=1,2,\cdots$, and for $n,m>N$. For a fixed j, $(\alpha_j^{(m)})_m$ is a Cauchy sequence in $\boldsymbol{C}$. Let $\beta_j=\lim_m \alpha_j^{(m)}$. Using this limits, we define $x=(\beta_1,\beta_2,\cdots)$ and show that $x\in l_2(q)$ and $\|x_n-x\|_q\to 0$ an $n\to\infty$.

From (*), we have $\left(\sum_{j=1}^{k}|\alpha_j^{(n)}-\alpha_j^{(m)}|^2q^{-j}\right)^{\frac{1}{2}}<\varepsilon$ for all $n,m>N$ ($k=1,2,3,\cdots$). Letting $m\to\infty$, we obtain $\left(\sum_{j=1}^{k}|\alpha_j^{(n)}-\beta_j|^2q^{-j}\right)^{\frac{1}{2}}\leqq\varepsilon$ for $n>N$, for $k=1,2,\cdots$.

Let $k\to\infty$. Then $\left(\sum_{j=1}^{\infty}|\alpha_j^{(n)}-\beta_j|^2q^{-j}\right)^{\frac{1}{2}}\leqq\varepsilon$ for $m>N$. This implies $x_n\to x$ and $x_n-x=(\alpha_j^{(n)}-\beta_j)\in l_2(q)$. Since $x_n\in l_2(q)$, we have $x=(x-x_n)+x_n\in l_2(q)$. Hence $l_2(q)$ forms a Banach space. It is enough to show that the norm satisfies the parallelogram law.

For $x=(\alpha_n)$ and $y=(\beta_n)$ in $l_2(q)$, $\|x+y\|_q^2+\|x-y\|_q^2=\sum_{j=1}^{\infty}|\alpha_j+\beta_j|^2q^{-j}+\sum_{j=1}^{\infty}|\alpha_j-\beta_j|^2q^{-j}=2\left(\sum_{j=1}^{\infty}|\alpha_j|^2q^{-j}+\sum_{j=1}^{\infty}|\beta_j|^2q^{-j}\right)=2\|x\|_q^2+2\|y\|_q^2$. Therefore, $l_2(q)$ becomes a Hilbert space.

LEMMA 2.4. *Let A be the left shift operator on $l_2(q)$. Then A is a bounded linear operator with $\|A\|=\sqrt{q}$ and $A^*=qV$, that is, $V^*=\frac{1}{q}A$, where V is a right shift operator on $l_2(q)$.*

Proof. Clearly A is a linear and $\|Ax\|_q^2=\sum_{j=1}^{\infty}|x_{j+1}|^2q^{-j}\leqq q\|x\|_q^2$ for all $x=(x_n)$ in $l_2(q)$. If $x_0=(0,q^2,0,0,\cdots)$, then $\|Ax_0\|_q^2=q\|x_0\|_q^2$. Hence A is a bounded linear operator with $\|A\|=\sqrt{q}$.

Define $\Omega: l_2(q)\times l_2(q)\to\boldsymbol{C}$ by $\Omega(x,y)=(x|Ay)$. Then Ω is a sesquilinear functional and

$$\|\Omega\|=\sup_{\substack{\|x\|_q\leq 1\\ \|y\|_q\leq 1}}\frac{|\Omega(x,y)|}{\|x\|_q\|y\|_q}=\sup_{\substack{\|x\|_q\leq 1\\ \|y\|_q\leq 1}}\frac{|(x|Ay)|}{\|x\|_q\|y\|_q}\leqq\sup_{\|y\|_q\leq 1}\frac{\|Ay\|_q}{\|y\|_q}.$$

On the other hand,

$$\|\Omega\|=\sup_{\substack{\|x\|_q\leq 1\\ \|y\|_q\leq 1}}\frac{|(x|Ay)|}{\|x\|_q\|y\|_q}\geqq\sup_{\|y\|_q\leq 1}\frac{|(Ay|Ay)|}{\|Ay\|_q\|y\|_q}=\sup_{\|y\|_q\leq 1}\frac{\|Ay\|_q}{\|y\|_q}.$$

Hence we have $\|\Omega\|=\|A\|=\sqrt{q}$.

For each $x\in l_2(q)$, defining $F_x: l_2(q)\to\boldsymbol{C}$ by $F_x(y)=\Omega(x,y)$. Then F_x is a bounded linear functional. We put $G=\{y\in l_2(q):\ F_x(y)=0\}$, then G is a closed linear subspace of $l_2(q)$. If $G=l_2(q)$, then clearly $F_x=0$; we choose $h=(0,0,0,\cdots)$, then $h\in l_2(q)$ and $\|F_x\|=\|h\|_q=0$. If $G\subsetneqq l_2(q)$, then there exists a non-zero element $h_0\in[l_2(q)\ominus G]$. Since $F_x(F_x(y)h_0-yF_x(h_0))=0$ for all $y\in l_2(q)$, $F_x(y)h_0-yF_x(h_0)\in G$ for all $y\in l_2(q)$. Moreover, $F_x(y)(h_0|h_0)-F_x(h_0)(y|h_0)=(F_x(y)h_0-yF_x(h_0)|h_0)=0$ for all $y\in l_2(q)$.

We put $h=\dfrac{\overline{F_x(h_0)}}{\|h_0\|_q^2}h_0$ then $F_x(y)=(y|h)$ for all $y\in l_2(q)$,

$\|h\|_q=\dfrac{|F_x(h_0)|}{\|h_0\|_q^2}\|h_0\|_q\leqq\|F_x\|$ and $\|F_x\|=$

$\sup_{\|y\|_q\leqq1}|F_x(y)|=\sup_{\|y\|_q\leqq1}|(y|h)|\leqq\|h\|_q$. Thus $\|F_x\|=\|h\|_q$.

Therefore for each $x\in l_2(q)$, there exists $h_x\in l_2(q)$ such that $F_x(y)=(y|h_x)$ and $\|F_x\|=\|h_x\|_q$. We define $A^*l_2(q)\to l_2(q)$ by $A^*x=h_x$. Then A^* is a bounded linear operator with $\|A^*\|=\|\Omega\|=\sqrt{q}$. Also $(Ay|x)=\Omega(x,y)=F_x(y)=(y|h_x)=(y|A^*x)$ for all $x,y\in l_2(q)$. Let $e_1=(q^{\frac{1}{2}}, 0,0,0,\cdots\cdots)$ and choose $h_0=(0,1,0,0,0,\cdots\cdots)$, then $h_0\notin\ker(F_{e_1})$ and $\|h_0\|_q^2=\dfrac{1}{q^2}$.

$$\text{Thus } h_{e_1}=\frac{(e_1|Ah_0)}{\|h\|_q^2}h_0=q(0,q^{\frac{1}{2}},0,0,\cdots\cdots)=qVe_1.$$

We have $A^*e_1=qVe_1$. Let $e_2=(0,q,0,0,0,\cdots\cdots)$. Choose $h_0=(0,0,1,0,0,\cdots\cdots)$, then $h_0\notin\ker(F_{e_2})$ and $\|h_0\|=\dfrac{1}{q^3}$.

Thus $h_{e_2}=qVe_2$, and so $A^*e_2=qVe_2$. Continuing this process, we have $A^*e_n=qVe_n$ for all n, where $\{e_n : e_n=(0,0,0,\cdots\cdots,q^{n/2},0,0,\cdots\cdots), n=1,2,3,\cdots\cdots\}$ is a complete orthonormal system of the $l_2(q)$. Hence $A^*=qV$ and $A=(A^*)^*=(qV)^*=qV^*$. Therefore we have $V^*=\dfrac{1}{q}A$. This completes the proof.

Now, we determine the spectra of the weight shift operators on the space $l_2(q)$ using the above Lemma 2.4. The results are as following.

THEOREM 2.5. *Suppose that* $0<|\alpha_1|\leqq|\alpha_2|\leqq|\alpha_3|\leqq\cdots$ *such that* $r=\sup|\alpha_n|<\infty$. *Let* $A: l_2(q)\to l_2(q)$ *be the operator defined by* $A(x_1,x_2,x_3,\cdots\cdots)=(\alpha_2x_2,\alpha_3x_3,\alpha_4x_4,\cdots\cdots)$.

Then we have the following:

(1) $\sigma_p(A)=\{\lambda\in\boldsymbol{C}:|\lambda|<r\sqrt{q}\}$ *and* $\sigma(A)=\sigma_{ap}(A)=\{\lambda\in\boldsymbol{C}:\ |\lambda|\leqq$

$r\sqrt{q}\}$,

(2) $\sigma_{com}(A)=\sigma_r(A)=\phi$ *and* $\sigma_c(A)=\{\lambda\in \boldsymbol{C}:\ |\lambda|=r\sqrt{q}\}$.

Proof. (1) Suppose that $Ax=\lambda x$ for $x=(x_n)$ in $l_2(q)$. Then $(\alpha_2x_2, \alpha_3x_3, \alpha_4x_4, \cdots\cdots)=(\lambda x_1, \lambda x_2, \lambda x_3, \cdots\cdots)$, that is, $\alpha_{n+1}x_{n+1}=\lambda x_n$ for all n. If $x_1=0$, then $x=0$ in $l_2(q)$. Let $\lambda\in\sigma_p(A)$. Then there exists a non-zero element $x=(x_n)$ such that $Ax=\lambda x$. Since $x_1\neq 0$, $\|\lambda x\|_q=\|Ax\|_q=\left(\sum_{j=2}^{\infty}|\alpha_jx_j|^2q^{-j+1}\right)^{\frac{1}{2}}<r\sqrt{q}\|x\|_q$.

Hence $\sigma_p(A)\subset\{\lambda\in\boldsymbol{C}:\ |\lambda|<r\sqrt{q}\}$. Let $|\lambda|<r\sqrt{q}$. Choose $x_0=\left(\sqrt{q},\ \frac{\lambda\sqrt{q}}{\alpha_2},\ \frac{\lambda^2\sqrt{q}}{\alpha_2\alpha_2},\cdots\cdots\right)\in l_2(q)$, then $Ax_0=\lambda x_0$, that is, $\lambda\in\sigma_p(A)$, and so $\{\lambda\in\boldsymbol{C}:\ |\lambda|<\sqrt{q}\}\subset\sigma_p(A)$. We have $\sigma_p(A)=\{\lambda\in\boldsymbol{C}:\ |\lambda|<r\sqrt{q}\}$. Clearly $\sigma_p(A)\subset\sigma_{ap}(A)\subset\sigma(A)\subset\{\lambda\in\boldsymbol{C}:\ |\lambda|\leqq r\sqrt{q}\}$. Since $\sigma_{ap}(A)$ and $\sigma(A)$ are closed subsets of $\boldsymbol{C}$. Hence $\sigma(A)=\sigma_{ap}(A)=\{\lambda\in\boldsymbol{C}:\ |\lambda|\leqq r\sqrt{q}\}$.

(2) It is known that $\sigma_{com}(A)=(\sigma_p(A^*))^*=q(\sigma_p(V))^*$, where $V: l_2(q)\to l_2(q)$ is the operator dfined by $V(x_1, x_2, x_3, \cdots\cdots)=(0, \alpha_1x_1, \alpha_2x_2, \alpha_3x_3, \cdots\cdots)$. It is enough to show that $\sigma_p(V)=\phi$. Suppose that $Vx=\lambda x$ for $x=(x_n)$ in $l_2(q)$. Then $(0, \alpha_1x_1, \alpha_2x_2, \cdots\cdots)=(\lambda x_1, \lambda x_2, \lambda x_3, \cdots\cdots)$, that is, $\lambda x_1=0$ and $\alpha_nx_n=\lambda x_{n+1}$ for all n. Hence $x=0$ in $l_2(q)$. It follows that $\sigma_p(V)=\phi$. Also $\sigma_r(A)=\sigma_{com}(A)\backslash\sigma_p(A)=\phi$ and $\sigma_c(A)=\sigma(A)\backslash\{\sigma_{com}(A)\cup\sigma_p(A)\}=\{\lambda\in\boldsymbol{C}:\ |\lambda|=r\sqrt{q}\}$.

COROLLARY 2.6. *Let A be the left shift operator on $l_2(q)$. Then*

(1) $\sigma_p(A)=\{\lambda\in\boldsymbol{C}: |\lambda|<\sqrt{q}\}$ *and* $\sigma(A)=\sigma_{ap}(A)=\{\lambda\in\boldsymbol{C}: |\lambda|\leqq\sqrt{q}\}$,

(2) $\sigma_{com}(A)=\sigma_r(A)=\phi$ *and* $\sigma_c(A)=\{\lambda\in\boldsymbol{C}:\ |\lambda|=\sqrt{q}\}$.

COROLLARY 2.7. *Let A be the left shift operator on* $l^2=\left\{x=(x_n)\ :\ \sum_{n=1}^{\infty}|x_n|^2<\infty\right\}$. *Then*

(1) $\sigma_p(A)=\{\lambda\in\boldsymbol{C}:\ |\lambda|<1\}$ *and* $\sigma(A)=\sigma_{ap}(A)=\{\lambda\in\boldsymbol{C}:\ |\lambda|\leqq 1\}$,

(2) $\sigma_{com}(A)=\sigma_r(A)=\phi$ *and* $\sigma_c(A)=\{\lambda\in\boldsymbol{C}:\ |\lambda|=1\}$.

THEOREM 2.8. *Let (α_n) be as in Theorem 2.5, and let $V: l_2(q)\to l_2(q)$ be the operator defined by $V(x_1, x_2, x_3, \cdots\cdots)=(0, \alpha_1x_1, \alpha_2x_2, \alpha_3x_3, \cdots\cdots)$. Then*

(1) $\sigma_p(V)=\phi$, $\sigma_{com}(V)=\sigma_r(V)=\left\{\lambda\in\boldsymbol{C}:\ |\lambda|<\frac{r}{\sqrt{q}}\right\}$ *and* $\sigma(V)=$

$\left\{\lambda\in \boldsymbol{C}:\ |\lambda|\leqq\frac{r}{\sqrt{q}}\right\}$.

(2) $\sigma_{ap}(V)=\sigma_c(V)=\left\{\lambda\in \boldsymbol{C}:\ |\lambda|=\frac{r}{\sqrt{q}}\right\}$.

Proof. (1) Clearly V is a bounded linear operator with $\|V\|\leqq\frac{r}{\sqrt{q}}$. By Theorem 2.5, $\sigma_p(V)=\phi$. It is known that $\sigma_{com}(V)=(\sigma_p((V^*)^*$ $=\frac{1}{q}(\sigma_p(A))^*$, where $A(x_1, x_2, x_3, \cdots\cdots)=(\alpha_2x_2, \alpha_3x_3, \cdots\cdots)$. Since $\sigma_p(A)$ $=\left\{\lambda\in \boldsymbol{C}:\ |\lambda|<r\sqrt{q}\right\}$, $\sigma_{com}(V)=\left\{\lambda\in \boldsymbol{C}:\ |\lambda|<\frac{r}{\sqrt{q}}\right\}$.

It follows that $\sigma_r(V)=\sigma_{com}(V)\backslash\sigma_p(V)=\sigma_{com}(V)$. Since $\sigma_{com}(V)\subset$ $\sigma(V)\subset\left\{\lambda\in \boldsymbol{C}:\ |\lambda|\leqq\frac{r}{\sqrt{q}}\right\}$ and $\sigma(V)$ is a closed subset of $\boldsymbol{C}$, we have $\sigma(V)=\left\{\lambda\in \boldsymbol{C}:\ |\lambda|\leqq\frac{r}{\sqrt{q}}\right\}$.

For (2), it is well-known that $\left\{\lambda\in \boldsymbol{C}:\ |\lambda|=\frac{r}{\sqrt{q}}\right\}=\partial\sigma(V)\subset\sigma_{ap}(V)$ $\subset\sigma(V)$. It is enough to show that $\sigma_{ap}(V)\subset\{\lambda\in \boldsymbol{C}:\ \frac{r}{\sqrt{q}}\leqq|\lambda|\}$. For a λ with $|\lambda|<\frac{r}{\sqrt{q}}$, $\|(V-\lambda I)x\|_q\geqq|\ \|Vx\|_q-|\lambda|\ \|x\|_q|\geqq|\frac{m}{\sqrt{q}}-|\lambda||\,\|x\|_q$ for all $x\in l_2(q)$, where $0<m<|\alpha_1|$ and $\frac{m}{\sqrt{q}}\neq|\lambda|$. Thus $V-\lambda I$ is bounded below, that is; $\lambda\notin\sigma_{ap}(V)$. Hence $\sigma_{ap}(V)\subset\left\{\lambda\in \boldsymbol{C}:\ \frac{r}{\sqrt{q}}\leqq|\lambda|\right\}$. It follows that $\sigma_{ap}(V)=\left\{\lambda\in \boldsymbol{C}:\ |\lambda|=\frac{r}{\sqrt{q}}\right\}$. Also we have $\sigma_c(V)=$ $\sigma(V)\backslash\{\sigma_{com}(V)\cup\sigma_p(V)\}=\left\{\lambda\in \boldsymbol{C}:\ |\lambda|=\frac{r}{\sqrt{q}}\right\}$.

From the Theorem 2.8, we have the following immediate consequences.

COROLLARY 2.9. *Let V be the right shift operator on $l_2(q)$. Then* $\sigma_p(V)=\phi$, $\sigma_{com}(V)=\sigma_r(V)=\left\{\lambda\in \boldsymbol{C}:\ |\lambda|<\frac{1}{\sqrt{q}}\right\}$, $\sigma(V)=\left\{\lambda\in \boldsymbol{C}:\ |\lambda|\leqq\frac{1}{\sqrt{q}}\right\}$ *and* $\sigma_{ap}(V)=\sigma_c(V)=\left\{\lambda\in \boldsymbol{C}:\ |\lambda|=\frac{1}{\sqrt{q}}\right\}$.

COROLLARY 2.10. *Let V be the right shift operator on l^2. Then we have $\sigma_p(V)=\phi$, $\sigma_{com}(V)=\sigma_r(V)=\{\lambda\in \boldsymbol{C}: |\lambda|<1\}$, $\sigma(V)=\{\lambda\in \boldsymbol{C}: |\lambda|\leqq 1\}$ and $\sigma_{ap}(V)=\sigma_c(V)=\{\lambda\in \boldsymbol{C}: |\lambda|=1\}$.*

By the calculation, the matrices representations of the resolvent operators $(\lambda I-A)^{-1}$, $(\lambda I-V)^{-1}$ have the following forms.

$$(\lambda I-A)^{-1}=\begin{pmatrix} \lambda^{-1} & \alpha_2\lambda^{-2} & \alpha_2\alpha_3\lambda^{-3} & \alpha_2\alpha_3\alpha_4\lambda^{-4}\cdots\cdots \\ 0 & \lambda^{-1} & \alpha_3\lambda^{-2} & \alpha_3\alpha_4\lambda^{-3}\cdots\cdots \\ 0 & 0 & \lambda^{-1} & \alpha_4\lambda^{-2}\cdots\cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$(\lambda I-V)^{-1}=\begin{pmatrix} \lambda^{-1} & 0 & 0 & 0\ \cdots\cdots \\ \alpha_1\lambda^{-2} & \lambda^{-1} & 0 & 0\ \cdots\cdots \\ \alpha_1\alpha_2\lambda^{-3} & \alpha_2\lambda^{-2} & \lambda^{-1} & 0\ \cdots\cdots \\ \alpha_1\alpha_2\alpha_3\lambda^{-4} & \alpha_2\alpha_3\lambda^{-3} & \alpha_3\lambda^{-2} & \lambda^{-1}\cdots\cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

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