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EXTENSIONS OF HIGHER ANTI-DERIVATIONS TO MODULES OF QUOTIENTS

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1. Introduction

Throughout the following, R will denote an associative ring with unit element 1 and R-Mod will denote the category of all unitary left R-modules. And let $w: R \longrightarrow R$ be an involution (i.e. w is an endomorphism of R whose square is identity map.) Then anti-derivation with respect to w of R is a mapping $d: R \longrightarrow R$ such that $d(a+b)=d(a)+d(b)$ and $d(ab)=d(a)b+w(a)d(b)$ for all elements $a, b \in R$ ([4]). If w is an identity map, then d is called an ordinary derivation. If M is a unitary left R-module and if d is a fixed anti-derivation (with respect to w) on R then anti-d-derivation on M is a mapping $\bar{d}: M \longrightarrow M$ satisfying the condition that $\bar{d}(m+n)=\bar{d}(m)+\bar{d}(n)$ and $\bar{d}(am)=d(a)m+w(a)\bar{d}(m)$ for all elements $m, n \in M$ and $a \in R$. If w is an identity map, then $\bar{d}$ is called a d-derivation on M ([3]).

Let S be a segment of N, i.e. $S=\{0, 1, 2, \cdots, s\}$ for some $s \geqq 0$. A family $d=(d_n)_{n \in s}$ of mappings $d_n: R \longrightarrow R$ is called anti-d-derivation of order s of R (where, $s=\sup S \leqq \infty$) if the following properties are satisfied (i) $d_n(a+b)=d_n(a)+d_n(b)$ (ii) $d_n(ab)=d_n(a)b+\sum_{i+j=n-1} d_i(a)d_j(b)+w(a)d_n(b)$ for all $a, b \in R$ (iii) $d_0=$identity map on R ([4, 5]).

If d is a fixed anti-d-derivation of order s on R, then anti-d-derivation of order s on M is a family $\bar{d}=(\bar{d}_n)_{n \in S}$ of mappings satisfying that (i) $d_n(m+m')=d_n(m)+d_n(m')$ (ii) $\bar{d}_n(am)=d_n(a)m+\sum_{i+j=n-1} d_i(a)\bar{d}_j(m)+w(a)\bar{d}_n(m)$ for all $a \in R$ and $m, m' \in M$ (iii) $\bar{d}_0=$identity map on M ([3]).

LEMMA 1. ([4, 5]) *The set of ordinary derivations of R corresponds bijectively to the set of derivations of order* 1 *of R. And the set of der-*

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ivations of order infinite corresponds bijectively to the inverse limit of the set of derivations of finite orders.

2. Preliminaries

Notations and terminology concerning (hereditary) torsion theories on R–Mod will follow [2]. In particular, if τ is a torsion theory on R–Mod then a left ideal H of R is said to be τ–dense in R if and only if the cyclic left R–module R/H is τ–torsion. If M is a left R–module then we denote by $T_\tau(M)$ the unique largest submodule of M which is τ–torsion. If $E(M)$ is the injective hull of a left R–module M then we define the submodule $E_\tau(M)$ of $E(M)$ by $E_\tau(M)/M=T_\tau(E(M)/M)$. The module of quotients of M with respect to τ, denoted by $Q_\tau(M)$, is then defined to be $E_\tau(M/T_\tau(M))$. Note that, in particular, if M is τ–torsionfree then $Q_\tau(M)=E_\tau(M)$, and this is a left R–module containing M as a largest submodule. In general, we have a canonical R–homomorphism from M to $Q_\tau(M)$ obtained by composing the canonical surjection from M to $M/T_\tau(M)$ with the inclusion map into $Q.(M)$.

If R is the endomorphism ring of the left R–module $Q_\tau({}_RR)$ then $Q_\tau(M)$ is canonically a left R–module for every R–module and the canonical map $R\longrightarrow R_\tau$ is a ring homomorphism, the ring R_τ is called as the ring of quotients or localization of R at τ. A torsion theory on R–Mod is said to be faithful if and only if R, considered as a left module over itself, is τ–torsionfree. In this case, R is canonically subring of R_τ.

Before entering our discussion, we assume that any anti–derivations are related with a fixed involution w.

LEMMA 2. ([2]) *Let H be a τ–dense ideal in R, and let $\alpha_{H,q}$ be R–module homomorphism defined on H into $Q_\tau(M)$, then R/H is τ–torsion and there exist unique R–module homomorphism $\beta_{R,q}:R\longrightarrow Q_\tau(M)$ which makes the diagram*

$$\begin{array}{ccccc} 0 & \longrightarrow & H & \longrightarrow & R \\ & & {\scriptstyle \alpha_{H,q}}\downarrow & \swarrow {\scriptstyle \beta_{R,q}} & \\ & & Q_\tau(M) & & \end{array}$$

commutes.

LEMMA 3. ([2]) *Let H and K be τ-dense ideals of R then we have the following results.*

(1) *$H \cap K$ is τ-dense ideal.*

(2) *$(H : a) = \{r \in R \mid ra \in H\}$ is τ-dense ideal.*

(3) *Homomorphic image of H is τ-dense ideal.*

LEMMA 4. ([2]) *Let H and K be τ-dense ideals of R and let $\alpha_{H,q}$: $H \longrightarrow Q_\tau(M)$ and $\alpha_{K,q} : K \longrightarrow Q_\tau(M)$ be defined as in the Lemma 2. Then $\alpha_{H,q}$ and $\alpha_{K,q}$ define the same element in $Q_\tau(M)$.*

3. Extension theorems

In this section we consider extensions of higher anti-d-derivation to modules of quotients, in the case module M is τ-torsionfree left R-module, where τ is a torsion theory on R-mod. We begin with a Lemma.

LEMMA 5. *For each q in $Q_\tau(M)$, the map $\alpha_{H,q} : H \longrightarrow Q_\tau(M)$ defined by $h \longrightarrow \bar{d}_n(w(h)q) - \sum_{i+j=n-1} d_i'(w(h))\bar{\bar{d}}_j'(q) - d_n(w(h))q$ is an R-module homomorphism for every $h \in H$, where d_i' is a derivation of order i on R and $\bar{\bar{d}}_j'$ is derivation of order j on $Q_\tau(M)$ which restricts to M is $\bar{d}_j$. Moreover the map defined by $k \longrightarrow (k)\alpha_{K,aq} - (kw(a))\alpha_{K,q}$ is an R-module homomorphism.*

Proof. The proof is routine use the definition of higher anti-d-derivation and higher derivation.

THEOREM 6. *Let d be an anti-d-derivation of order s on R and let τ be a torsion theory on R-Mod and M be τ-torsionfree left R-module on which we have defined an anti-d-derivation $\bar{d}$ of order s. Then there exists an anti-d-derivation of order s, $\bar{\bar{d}}$ defined on $Q_\tau(M)$, the restriction of which to M is $\bar{d}$.*

Proof. In the case of finite order, we use the mathematical induction on the order s. For $s=0$, the statement is trivial. For $s=1$, if $q \in Q_\tau(M)$, then there exists a τ-dense left ideal H of R satisfying $Hq \subseteq M$. Define a function $\alpha_{H,q} : H \longrightarrow Q_\tau(M)$ by setting $h \longrightarrow \bar{d}(w(h)q) - d(w(h))q$, by the Lemma 2 we see that $\alpha_{H,q}$ extends uniquely to R-homomorphism from ${}_R R$ to $Q_\tau(M)$ and so there exists unique element $\bar{\bar{q}}$ of $Q_\tau(M)$ satisfying the condition that $\bar{\bar{d}}(q) = \bar{\bar{q}}$. This function is well-defined and becomes anti-d-derivation of order 1, moreover restricts to M is $\bar{d}$.

Assume that for the case of $s=n-1$, the statement is true. If q is an element of $Q_\tau(M)$ then there exists τ-dense left ideal H of R satisfying $Hq\subseteq M$ and $w(H)q\subseteq M$. Let $\alpha_{H,q}$ be as in the Lemma 5, then by the Lemma 2, we see that $\alpha_{H,q}$ extends uniquely to R-homomorphism from ${}_RR$ to $Q_\tau(M)$ and so there exists unique element $\bar{\bar{q}}\in Q_\tau(M)$ satisfying the condition that $(h)\alpha_{H,q}=h\bar{\bar{q}}$ for all element $h\in H$. We define a function $\bar{\bar{d}}_n : Q_\tau(M)\longrightarrow Q_\tau(M)$ by setting $\bar{\bar{d}}_n(q)=\bar{\bar{q}}$. This function is well-defined. Indeed, suppose that q is an element of $Q_\tau(M)$ and let H and K be τ-dense left ideals of R satisfying $Hq\subseteq M$ and $Kq\subseteq M$. Then $(H\cap K)q\subseteq M$ and $H\cap K$ is τ-dense left ideal of R, by the Lemma 4 $\alpha_{H,q}$ and $\alpha_{K,q}$ define the same element $\bar{\bar{q}}$.

Now we claim that such $\bar{\bar{d}}_n$ is anti-d-derivation of order n on $Q_\tau(M)$. Indeed, let q and q' be elements of $Q_\tau(M)$ and let a be an element of R, then there exist τ-dense left ideals H and H' of R satisfying $Hq\subseteq M$ and $H'q'\subseteq M$. Take $K=H\cap H'$, then we have $Kq\subseteq M$ and $Kq'\subseteq M$, so $K(q+q')\subseteq M$. Moıeover, for each element $k\in K$ we have

$$\begin{aligned}(k)\alpha_{K,q+q'}&=\bar{d}_n(w(k)(q+q'))-\sum_{i+j=n-1}d_i'(w(k))d_j'(q+q')\\&\quad-d_n(w(k))(q+q')\\&=(k)\alpha_{K,q}+(k)\alpha_{K,q'}\\&=(k)(\alpha_{K,q}+\alpha_{K,q'}).\end{aligned}$$

By the Lemma 2, the uniqueness of extension, this implies that $\bar{\bar{d}}_n(q+q')=\bar{\bar{d}}_n(q)+\bar{\bar{d}}_n(q')$. Similary there exists a τ-dense left ideal H of R satisfying conditions that $Hq\subseteq M$, $Haq\subseteq M$, $w(H)q\subseteq M$ and $w(H)aq\subseteq M$, let $K=H\cap w(H)\cap(H;a)\cap(w(H):a)$, by the Lemma 3, K is a τ-dense left ideal of R, we therefore have an R-homomorphism from ${}_RK$ to $Q_\tau(M)$, which can be extended to from ${}_RR$ into $Q_\tau(M)$. We see that $(k)\alpha_{K,aq}-(kw(a))\alpha_{K,q}=\bar{d}_n(w(k)aq)-\sum_{i+j=n-1}\{d_i'(w(k))\bar{\bar{d}}_j'(aq)\}-d_n(w(k))aq-\bar{d}_n(w(k)aq)+\sum_{i+j=n-1}\{d_i'(w(k)a)\bar{\bar{d}}_j'(q)\}+d_n(w(k)a)q=-\sum_{i+j=n-1}\{d_i(w(k))\sum_{s+t=j}d_s(a)\bar{\bar{d}}_t(q)\}-d_n(w(k))aq+\sum_{i+j=n-1}\{\sum_{s+t=i}d_s(w(k))d_t(a)\bar{\bar{d}}_j(q)\}+d_n(w(k))aq+\sum_{s+t=n-1}\{d_s(w(k))d_t(a)\}+kd_n(a)q=kd_n(a)q+k\sum_{i+j=n-1}\{d_i(a)\bar{\bar{d}}_j(q)\}$, and so by the Lemma 2, this implies that $\bar{\bar{d}}_n(aq)=d_n(a)q+\sum_{i+j=n-1}\{d_i(a)\bar{\bar{d}}_j(q)\}+w(a)\bar{\bar{d}}_n(q)$, thus $\bar{\bar{d}}_n$ is an anti-d-derivation of order n on $Q_\tau(M)$.

Now we prove that $\bar{\bar{d}}$ restricts to $\bar{d}$ on M. Indeed, for every $m\in M$, then we take H equal to R itself and so we see that for any $a\in R$ we have $\bar{d}_n(am) - \sum_{i+j=n-1} d_i(a)\bar{d}_j(m) - d_n(a)m = w(a)\bar{d}_n(am)$, which implies that $\bar{\bar{d}}_n(am) = \bar{d}_n(am)$ for each $n \in S$.

In the case of infinite order, we use the Lemma 1 not only ring R, but also module M and $Q_\tau(M)$, i.e. for any infinite order (anti-d-) derivation d_∞ ($\bar{d}_\infty$ or $\bar{\bar{d}}_\infty$) on R (M or $Q_\tau(M)$), then there exists unique sequence (anti-d-) derivations d_n ($\bar{d}_n$ or $\bar{\bar{d}}_n$) on R (M or $Q_\tau(M)$) such that we can write $d_\infty = \varprojlim d_n$ ($\bar{d}_n = \varprojlim \bar{d}_n$ or $\bar{\bar{d}}_\infty = \varprojlim \bar{\bar{d}}_n$). For the given $\bar{d}_\infty$ there is unique sequence $\{\bar{d}_n\}_{n\in N}$ on M which we can write $\bar{d}_\infty = \varprojlim \bar{d}_n$, by the finite order case we can extend each $\bar{d}_n$ to $\bar{\bar{d}}_n$ on $Q_\tau(M)$ which restricts to $\bar{d}_n$ to M. Now take $\bar{\bar{d}}_\infty$ as an inverse limit of such $\{\bar{\bar{d}}_n\}_{n\in N}$ on $Q_\tau(M)$, then $\bar{\bar{d}}_\infty$ satisfies all results.

For the anti-d-derivations (of order 1) d on a ring R, then there exists a unique anti-d-derivation $\bar{\bar{d}}$ defined on R_τ, the restriction of which to R is d, in the case τ is a faithful torsion theory on R-Mod ([6]). Now we generalize this result to the higher order case.

THEOREM 7. *Let d be an anti-d-derivation of order s on R and let τ be a faithful torsion theory on R-Mod. Then there exists a unique anti-d-derivation $\bar{\bar{d}}$ of order s defined on R_τ, the restriction of which to R is d.*

Proof. The existence of $\bar{\bar{d}}$ follows from the Theorem 6 and the fact that $Q_\tau(R)$ and R_τ are isomorphic, as left R-modules. To show uniqueness assume that d' and d'' be anti-d-derivations of order s defined on R_τ and $d' = d''$ on R. For any non zero element $q \in R_\tau$ there is a τ-dense left ideal H of R satisfying conditions $Hq \subseteq R$ and $w(H)q \subseteq R$, take $K = H \cap w(H)$ as τ-dense ideal of R, then for any element $k \in K$ we have $0 = (d_n' - d_n'')(kq) = w(k)(d_n' - d_n'')(q)$, for each $n \in S$. Thus we have $w(K)(d_n' - d_n'')(q) = 0$, for each $n \in S$. Since $w(K)$ is a τ-dense ideal of R, this implies that $d_n'(q) = d_n''(q)$ for all $q \in R_\tau$.

COROLLARY 8. *Let d be an anti-d-derivation of order s on R and $\bar{d}$ be anti-d-derivation of order s on a left R-module M. Suppose that τ is a torsion theory on R-Mod satisfying the condition, for each $n \in S$, $\bar{d}_n(T_\tau(M)) \subseteq T_\tau(M)$. Then there exist an anti-d-derivation $\bar{\bar{d}}$ of order s on $Q_\tau(M)$ in such manner that the diagram*

$$\begin{array}{ccc} M & \longrightarrow & Q_\tau(M) \\ \bar{d}\downarrow & & \downarrow\bar{\bar{d}} \\ M & \longrightarrow & Q_\tau(M) \end{array}$$

commutes.

Proof. Define d' on $M/T_\tau(M)$ by denotting for each $n\in S$, $d_n' : m + T_\tau(M) \longrightarrow \bar{d}_n(m) + T_\tau(M)$, by the condition $\bar{d}_n(T_\tau(M)) \subseteq T_\tau(M)$, such a map is well-defined. And $M/T_\tau(M)$ is τ-torsionfree left R-module, by the Theorem 6, this derivation d' can be extended to anti-d-derivation $\bar{\bar{d}}$ on $Q_\tau(M)$ making the diagram commutes.

Now we consider inner derivation of order s on R, if there exists an element $\alpha=(a_n)_{n\in S}\in R\times R\times\cdots\times R$ ($s+1$-times) such that $d=\Delta(\alpha)$, where $d_1(x)=\Delta(\alpha)_1(x)=a_1x-xa_1$, $d_2(x)=\Delta(\alpha)_2=a_1{}^2x-a_1xa_1+a_2x-xa_2$, $d_3(x)=\Delta(\alpha)_3(x)=a_1{}^3x-a_1{}^2xa_1+a_1a_2x+xa_2a_1-a_1xa_2-a_2xa_1+a_3x-xa_3, \cdots$, we call d as an inner derivation of order s of R. ([1, 4])

COROLLARY 9. *The extension of any inner derivation d of order s of R to a derivation $\bar{\bar{d}}$ on R_τ is again inner. In particular, if τ is torsion-free, such extension $\bar{\bar{d}}$ is unique and which restricts to d on R.*

Proof. Let d be any inner derivation of order s on R, then there exists a sequence $\alpha=(a_n)_{n\in S}$ such that $d=\Delta(\alpha)$. Since R is τ-torsionfree $T_\tau(R)=0$, so for each $n\in S$ $d_n(T_\tau(R))=0\subseteq T_\tau(R)$. Take w=identity map on R in the Corollary 8, there exists an exension $\bar{\bar{d}}$ on $Q_\tau(R)$, so we can define a derivation $\bar{\bar{d}}$ on $Q_\tau(R)$ for the element $\alpha=(a_n)_{n\in S}$ as follows $\bar{\bar{d}}(q)=\Delta(\alpha)(q)$, then $\bar{\bar{d}}$ is an inner derivation of order s. On the other hand τ is faithful, by the Theorem 7, such extension is unique and which restricts d on R.

If we take $S=\{0, 1\}$, by the Lemma 1 we have following Corollary.

COROLLARY 10. *If $l_a : R \longrightarrow R$ is the inner derivation of R defined by an element a and if τ is a faithful torsion theory on R-Mod then a defines an inner derivation $\bar{\bar{l}}_a$ on R_τ which restricts to l_a on R.* ([3]).

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