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HERMITIAN ELEMENTS OF A BANACH ALGEBRA

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1. Introduction

Throughout this paper, A denotes a complex unital Banach algebra. An element h of A is Hermitian if its numerical range is real. Let H be the set of all Hermitian elements of A. This paper deals with the following question; If $a, b, ab \in H$, does it then follow that $ab=ba$?

Berkson [1] has proved various partial positive results, one is that, if a, b, ab, a^2 and b^2 are all Hermitian, then $ab=ba$. Murphy [4] extended Berkson's result, that is, if a, b and ab are Hermitian and also either a^2 or b^2 is Hermitian, then $ab=ba$.

2. Main results

The following three lemmas contain the elementary properties of the Hermitian elements, which can be found in [2].

LEMMA 2.1. (1) *H is a real linear subspace of A.* (2) $H \cap iH = \{0\}$.

LEMMA 2.2. *If $h, k \in H$, then $i(hk-kh) \in H$.*

LEMMA 2.3. (*Sinclair's Theorem*) *If $h \in H$, then $r(h)=\|h\|$, where r denotes spectral radius.*

We use the following lemma, which was proved by Kleinecke [3].

LEMMA 2.4. *Let B be a Banach algebra. Let $x, y \in B$. Let x commute with $xy-yx$. Then $xy-yx$ is quasinilpotent, that is, $r(xy-yx)=0$.*

Now we have the main theorem.

THEOREM 2.5. *Let $a, b, ab \in H$. Suppose also that either $i(bxb-xb^2) \in H$ and $a^2+xb \in H$ for some x in A or $i(aya-ya^2) \in H$ and $b^2+ya \in H$ for some y in A. Then $ab=ba$.*

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Proof. Suppose first that $i(bxb-xb^2)\in H$ and $a^2+xb\in H$ for some x in A.

Apply Lemma 2.2 with $h=a$, $k=b$. So

$$i(ab-ba)\in H. \tag{1}$$

Apply Lemma 2.2 with $h=-a$, $k=i(ab-ba)$. So

$$a(ab-ba)-(ab-ba)a\in H. \tag{2}$$

Apply Lemma 2.2 with $h=a$, $k=ab$. So

$$i(a^2b-aba)\in H. \tag{3}$$

Apply Lemma 2.2 with $h=b$, $k=a^2+xb$. So

$$i(ba^2-a^2b+bxb-xb^2)\in H.$$

Since $i(bxb-xb^2)\in H$,

$$i(ba^2-a^2b)\in H. \tag{4}$$

Taking twice (3) plus (4), we conclude that

$$i(a^2b-2aba+ba^2)\in H.$$

$$\text{i.e. } i(a(ab-ba)-(ab-ba)a)\in H. \tag{5}$$

Apply Lemma 2.1(2) to (2) and (5) to deduce that

$$a(ab-ba)=(ab-ba)a.$$

Hence, by Lemma 2.4 $ab-ba$ is quasinilpotent. So, by (1), $i(ab-ba)$ is both Hermitian and quasinilpotent. Sinclair's Theorem then applies to $i(ab-ba)$ to prove that $ab=ba$.

The same conclusion follows when $i(aya-ya^2)$ and b^2+ya for some y in A are Hermitian by considering A with its multiplication reversed.

COROLLARY 2.6. *Let $a, b, ab\in H$. Suppose also that either $a^2+rb^n\in H$ for some real number r and positive integer n or $b^2+sa^n\in H$ for some real number s and positive integer n. Then $ab=ba$.*

Proof. Apply Theorem 2.5 to $x=rb^{n-1}$ and $y=sa^{n-1}$.

We obtain Murphy's Theorem as a corollary.

COROLLARY 2.7. (*Murphy's Theorem*) *Let $a, b, ab\in H$. Suppose also that either $a^2\in H$ or $b^2\in H$. Then $ab=ba$.*

Proof. Apply Corollary 2.6 to $r=0$ and $s=0$.

COROLLARY 2.8. *Let $a, b, ab\in H$. Suppose also that either $a^2+xb\in H$, $bx=xb$ for some x in A or $b^2+ya\in H$, $ay=ya$ for some y in A. Then $ab=ba$.*

Proof. It is trivial by Theorem 2.5.

References

1. E. Berkson, *Hermitian projections and orthogonality in Banach spaces*, Proc. London Math. Soc., **24**(1972), 101–118.
2. F.F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and elements of normed algebras* (Cambridge University Press, 1971).
3. D.C. Kleinecke, *On operator commutators*, Proc. Amer. Math. Soc., **8** (1957), 535–536.
4. I.S. Murphy, *A note on Hermitian elements of a Banach algebra*, J. London Math. Soc. (3), **6**(1973), 427–428.

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