ON WC-CONTINUOUS FUNCTIONS (*)

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1. Introduction

In 1970, Gentry and Hoyle [5] defined a function $f: X \rightarrow Y$ to be C-continuous if for each $x \in X$ and each open set V containing f(x) and having the compact complement, there exists an open set U containing x such that $f(U) \subset V$. These functions have been investigated by Long and Hendrix [6] and Long and Herrington [8]. In 1980, Long and Hamlett [7] called a function H-continuous by replacing "compact" in the definition of C-continuous functions with "H-closed" (quasi H-closed relative to Y [10]). The investigation of H-continuous functions has been continued by the second author [9] of the present paper.

Recently, Lo Faro and the first author [1,2] have introduced and investigated a new weak form of compactness in topological spaces, called weakly compact spaces. In this paper, we introduce and characterize sets called weakly compact relative to a topological space. Then we define a new class of functions called WC-continuous functions analogous to H-continuous functions and C-continuous functions. It will be shown that WC-continuity implies H-continuity and they are equivalent if the range of the function is almost-regular [11].

2. Definitions

Throughout this paper X and Y represent topological spaces on which no separation axioms are assumed unless explicitly stated. Let S be a subset of a space X. The closure and the interior of S in X are deno-

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ted by $Cl_X(S)$ and $Int_X(S)$ (or simply Cl(S) and Int(S)), respectively. A subset S is said to be regular open (resp. regular closed) if Int(Cl(S)) = S(resp. Cl(Int(S)) = S). For definitions and notations used in this paper, readers can find them in [2] except for the following.

DEFINITION 2.1. An open cover $\{V_{\alpha} | \alpha \in \mathcal{V}\}$ of a space X is said to be regular [2] if for each $\alpha \in \mathcal{V}$ there exists a nonempty regular closed set F_{α} in X such that $F_{\alpha} \subset V_{\alpha}$ and $X = \bigcup \{ Int(F_{\alpha}) | \alpha \in \mathcal{V} \}$.

DEFINITION 2.2. A space X is said to be weakly compact [2] (resp. quasi H-closed [10]) if every regular (resp. open) cover of X has a finite subfamily whose closure is a cover of X.

In [12], Singal and Singal called quasi *H*-closed spaces almost compact. A quasi *H*-closed Hausdorff space is usually called *H*-closed. Urysohn-closed spaces are characterized by weakly compact Urysohn spaces [3]. It has been shown in [2] that almost compactness is strictly stronger than weak compactness.

DEFINITION 2.3. A space X is said to be almost-regular [11] if for each regular closed set F of X and each point $x \in X - F$, there exist disjoint open sets U and V such that $F \subset U$ and $x \in V$.

DEFINITION 2.4. A subset K of a space X is said to be weakly compact relative to X if for each cover $\{V_{\alpha} | \alpha \in V\}$ of K by open sets of X satisfying the following property (P), there exists a finite subset V_0 of V such that $K \subset \bigcup \{\operatorname{Cl}_X(V_{\alpha}) | \alpha \in V_0\}$.

(P) For each $\alpha \in V$, V_{α} contains a nonempty regular closed set F_{α} of X and $K \subset \bigcup \{ \operatorname{Int}_{X}(F_{\alpha}) \mid \alpha \in V \}$.

DEFINITION 2.5. Let $\mathcal F$ be a filter on a space X. A point $x \in X$ is called a γ -adherence point of $\mathcal F$ [2] if $\mathcal F \wedge \mathcal U(\bar{\mathcal U}_X) \neq 0$.

DEFINITION 2.6. Let A be a subset of a space X. A point $x \in X$ is called a γ -adherence point of A if $A \cap V \neq \phi$ for every $V \in \mathcal{U}(\bar{\mathcal{U}}_X)$. The set of all γ -adherence points of A is called the γ -closure of A. If A contains the γ -closure of A, then it is called γ -closed.

3. Sets weakly compact relative to a space

DEFINITION 3.1. A filter \mathcal{F} on a space X is said to be quasi-regular [2] if there exists an open filter \mathcal{G} on X such that $\mathcal{F}=\mathcal{U}(\overline{\mathcal{G}})$.

REMARK 3.2. It is obvious that for any subset A of a space X $\operatorname{tr}_A \mathcal{F} \neq \mathcal{O}$ if $\operatorname{tr}_A \mathcal{G} \neq \mathcal{O}$, where $\operatorname{tr}_A \mathcal{F}$ denotes the trace of \mathcal{F} on A. However, the converse is not true in general as the following example shows.

EXAMPLE 3. 3. Let $X = \{x, y, z, t\}$, $\tau = \{\phi, X, \{x\}, \{z\}, \{z, t\}, \{x, z\}, \{x, y, z\}, \{x, y, z\}, \{x, z, t\}\}$ and $A = \{z, t\}$. Let $Q = \overline{\{x\}}$. Then the filter $\mathcal{F} = \mathcal{U}(\overline{Q}) = \{\{x, y, z\}, X\}$ is quasi-regular [2, Controesempio 4]. Moreover, $\operatorname{tr}_A \mathcal{F} = \{\{z\}, A\} \neq \mathcal{O}$ but $\operatorname{tr}_A \mathcal{Q} = \mathcal{O}$ because $\overline{\{x\}} \cap \{z, t\} = \phi$.

THEOREM 3.4. For a subset A of a space X, the following are equivalent:

- (1) Λ is weakly compact relative to X.
- (2) Every open filter Q with $tr_AQ \neq 0$ has a γ -adherence point in A.
- (3) Every filter \overline{Q} such that Q is an open filter and $tr_AQ \neq 0$ has an r-adherence point in A.
- (4) Every quasi-regular filter $\mathcal{F}=\mathcal{U}(\bar{Q})$ such that $tr_AQ\neq 0$ has an adherence $(\delta$ -adherence or r-adherence) point in A.
- (5) Every filter $\overline{\mathcal{F}}$ such that \mathcal{F} is a quasi-regular filter $\mathcal{F} = \mathcal{U}(\overline{\mathcal{Q}})$ with $tr_A \mathcal{Q} \neq 0$ has an adherence (δ -adherence) point in A.
- (6) Every filter $\bar{\mathcal{F}}$ such that \mathcal{F} is a quasi-regular filter $\mathcal{F} = \mathcal{U}(\bar{\mathbb{Q}})$ with $tr_A \mathbb{Q} \neq 0$ has an adherence (δ -adherence or r-adherence) point in A.
 - (7) Every open ultra filter Q with $tr_AQ \neq 0$ r-converges.
- (8) Let $\{C_{\alpha} | \alpha \in V\}$ be a family of closed sets of X such that for each $\alpha \in V$ there exists an open set A_{α} of X satisfying $C_{\alpha} \subseteq A_{\alpha}$ and $\bigcap \{Cl(A_{\alpha}) | \alpha \in V\} \subseteq X A$. Then there exists a finite subset V_0 of V such that $\bigcap \{Int(C_{\alpha}) | \alpha \in V_0\} \subseteq X A$.

Proof. (1) \Rightarrow (2): Let \mathscr{Q} be an open filter on X with $\operatorname{tr}_A \mathscr{Q} \neq \emptyset$. We suppose that $\mathscr{Q} \wedge \mathscr{U}(\bar{\mathcal{U}}_x) = \emptyset$ for every $x \in A$. Then, there exist open sets $G_x \in \mathscr{Q}$, $U_x \in \mathscr{U}_x$ and $A_x \in \mathscr{U}(\bar{\mathcal{U}}_x)$ such that $G_x \cap A_x = \phi$ and $U_x \subset \operatorname{Cl}(U_x) \subset A_x$. By $G_x \cap A_x = \phi$, we obtain $\operatorname{Cl}(G_x) \cap A_x = \phi$ and hence $\operatorname{Cl}(G_x) \cap \operatorname{Cl}(U_x) = \phi$. Let us put $B_x = X - \operatorname{Cl}(G_x)$, then $\operatorname{Cl}(U_x) \subset B_x$ and $B_x \in \mathscr{U}(\bar{\mathcal{U}}_x)$. The family $\{B_x \mid x \in A\}$ is a cover of A by open sets of X and $A \subset \bigcup \{\operatorname{Int}(\operatorname{Cl}(U_x)) \mid x \in A\}$. Therefore, there exists a finite number of points x_1, x_2, \cdots, x_n in A such that $A \subset \bigcup \{\operatorname{Cl}(B_{x_i}) \mid i = 1, 2, \cdots, n\}$. Therefore, we have

$$(*) \qquad \qquad \cap \{X - \operatorname{Cl}(B_{x_i}) \mid i = 1, 2, \dots, n\} \subset X - A.$$

For each $i=1, 2, \dots, n$, $G_{x_i} \subset \operatorname{Int}(\operatorname{Cl}(G_{x_i}))$, hence we have $X - \operatorname{Cl}(B_{x_i}) = \operatorname{Int}(X - B_{x_i}) = \operatorname{Int}(\operatorname{Cl}(G_{x_i})) \in \mathcal{G}$.

Therefore, by (*) we obtain $X-A \in \mathcal{Q}$. This is a contradiction.

- $(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (4) \Rightarrow (7) \Rightarrow (1)$: These implications are proved similarly to the proof of [2, Lemma 2.1].
- $(4) \Rightarrow (8)$: Let $\Gamma(V)$ be the family of all finite subsets of V. We suppose that

$$\cap \{ \operatorname{Int}(C_{\alpha}) \mid \alpha \in \Delta \} \not\subset X - A \text{ for every } \Delta \in \Gamma(\overline{V}).$$

Then, $\mathcal{F} = \{ \bigcap_{\alpha \in \mathcal{A}} \operatorname{Int}(C_{\alpha}) \mid \mathcal{A} \in \Gamma(\overline{V}) \}$ is an open filter base with $\operatorname{tr}_{A}\mathcal{F} \neq \mathcal{O}$. Thus, $\mathcal{U}(\overline{\mathcal{F}})$ is a quasi-regular filter on X such that $\operatorname{tr}_{A}\mathcal{F} \neq \mathcal{O}$. By (4), there exists a point $x \in A$ such that $\mathcal{U}(\overline{\mathcal{F}}) \wedge \mathcal{U}_{x} \neq \mathcal{O}$. Put

$$\mathcal{L} = \{ \bigcap_{\alpha \in \mathcal{A}} A_{\alpha} | \mathcal{\Delta} \in \Gamma(\mathcal{V}) \},$$

then it is an open filter base such that $\mathcal{U}(\bar{\mathcal{F}}) \subset \mathcal{L}$. Therefore, $\mathcal{L} \wedge \mathcal{U}_x \neq \mathcal{O}$ and hence $x \in \operatorname{Cl}(A_\alpha)$ for every $\alpha \in V$. Thus, we obtain $x \in \cap \{\operatorname{Cl}(A_\alpha) \mid \alpha \in V\}$. This is a contradiction because $\bigcap \{\operatorname{Cl}(A_\alpha) \mid \alpha \in V\} \subset X - A$.

 $(8) \Rightarrow (1)$: Let $\{A_{\alpha} | \alpha \in \mathcal{V}\}$ be an open cover of A with Property (P). For each $\alpha \in \mathcal{V}$, there exists a nonempty regular closed set C_{α} such that $C_{\alpha} \subseteq A_{\alpha}$ and $A \subseteq \bigcup \{ \operatorname{Int}(C_{\alpha}) | \alpha \in \mathcal{V} \}$. We consider the family $\{X - A_{\alpha} | \alpha \in \mathcal{V}\}$ of closed sets. For each $\alpha \in \mathcal{V}$, $X - C_{\alpha}$ is open in X, $X - C_{\alpha} \subseteq X - A_{\alpha}$ and

$$\cap \left\{ \operatorname{Cl}(X - C_{\alpha}) \mid \alpha \in \mathcal{V} \right\} = X - \cup \left\{ \operatorname{Int}(C_{\alpha}) \mid \alpha \in \mathcal{V} \right\} \subset X - A.$$

By (8), there exists a finite subset V_0 of V such that

$$\cap \{ \operatorname{Int}(X - A_{\alpha}) \mid \alpha \in V_0 \} \subset X - A.$$

Therefore, we obtain $A \subseteq \bigcup \{Cl(A_{\alpha}) \mid \alpha \in \mathcal{V}_0\}$. This shows that A is weakly compact relative to X.

4. WC-continuous functions

DEFINITION 4.1. A function $f: X \rightarrow Y$ is said to be WC-continuous if for each $x \in X$ and each open neighborhood V of f(x) having the complement weakly compact relative to Y, there exists an open neighborhood U of x such that $f(U) \subset V$.

THEOREM 4.2. For a function $f: X \rightarrow Y$ the following are equivalent:

- (1) f is WC-continuous.
- (2) If V is open in Y and Y-V is weakly compact relative to Y, then $f^{-1}(V)$ is open in X.
- (3) If F is closed in Y and weakly compact relative to Y, then $f^{-1}(F)$ is closed in X.

- *Proof.* (1) \Rightarrow (2): Let V be an open set of Y such that Y-V is weakly compact relative to Y. Let $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists an open neighborhood U of x such that $f(U) \subset V$. Therefore, we have $x \in U \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open in X.
 - $(2) \iff (3)$: This is obvious.
- $(3) \Longrightarrow (1)$: Let $x \in X$ and V an open neighborhood of f(x) such that Y V is weakly compact relative to Y. By (3), $f^{-1}(Y V)$ is closed in X and hence $U = f^{-1}(V)$ is an open set containing x such that $f(U) \subset V$.

LEMMA 4.3. If A_1 and A_2 are weakly compact relative to a space X, then $A_1 \cup A_2$ is weakly compact relative to X.

Proof. Let $\mathfrak{D} = \{V_{\alpha} | \alpha \in \mathcal{V}\}$ be a cover of $A_1 \cup A_2$ by open sets of X satisfying Property (P). Then \mathfrak{D} is a cover of A_1, A_2 satisfying (P) and hence for each i=1,2 there exists a finite subset \mathcal{V}_i of \mathcal{V} such that $A_i \subset \bigcup \{\operatorname{Cl}(V_{\alpha}) | \alpha \in \mathcal{V}_i\}$. Therefore, we have

$$A_1 \cup A_2 \subset \cup \{\operatorname{Cl}(V_\alpha) \mid \alpha \in V_1 \cup V_2\}.$$

This shows that $A_1 \cup A_2$ is weakly compact relative to X.

Let (X, τ) be a topological space. It follows from Lemma 4.3 that the family of open sets having the complement weakly compact relative to (X, τ) may be used as a base for a topology τ_{WC} . It has been shown that the family of open sets having the compact (resp. quasi H-closed) complement may be used as a base to generate a topology τ_C (resp. τ_H) on X [5, 7].

REMARK 4.4. For a topological space (X, τ) , we have $\tau_C \subset \tau_H \subset \tau_{WC} \subset \tau$.

THEOREM 4.5. A function $f: X \rightarrow (Y, \sigma)$ is WC-continuous if and only if $f: X \rightarrow (Y, \sigma_{WC})$ is continuous.

Proof. This is obvious from the definition of σ_{WC} .

REMARK 4.6. It is obvious that continuity implies WC-continuity and WC-continuity implies H-continuity. The following example shows that WC-continuity does not necessarily imply continuity.

EXAMPLE 4.7. Let X be the set of real numbers with the usual topology and $f: X \rightarrow X$ a function defined as follows: f(x) = 1/x if $x \neq 0$; f(0) = 1/2. Then f is C-continuous [5, Example 2] and by Theorem

4.17 (below) f is WC-continuous. However, f is not continuous.

For a function $f: X \rightarrow Y$, the set $\{(x, f(x)) | x \in X\}$ is called the *graph* of f and denoted by G(f).

THEOREM 4.8. If $f: X \rightarrow Y$ is an open function and G(f) is γ -closed in the product space $X \times Y$, then f is WC-continuous.

Proof. We suppose that f is not WC-continuous at some point $x \in X$. Then there exists an open set V containing f(x) and having the complement weakly compact relative to Y such that $f(U) \cap (Y - V) \neq \phi$ for every open set U containing x. Since f is open,

$$Q = \{ f(U) \mid x \in U \text{ and } U \text{ is open in } X \}$$

is an open filter base with $\operatorname{tr}_{Y-V} \mathcal{Q} \neq \mathcal{O}$. Since Y-V is weakly compact relative to Y, by (2) of Theorem 3.4 \mathcal{Q} has a γ -adherence point $y \in Y-V$. Therefore, $y \neq f(x)$ and (x, y) is a γ -adherence point of G(f). However, we have $(x, y) \notin G(f)$. This is a contradiction.

The following three theorems are immediate consequences of Theorem 4.5 and the proofs are omitted.

THEOREM 4.9. If $f: X \rightarrow Y$ is WC-continuous and A is a subset of X, then the restriction $f|A: A \rightarrow Y$ is WC-continuous.

THEOREM 4.10. If $f: X \rightarrow Y$ continuous and $g: Y \rightarrow Z$ is WC-continuous then the composition $g \circ f: X \rightarrow Z$ is WC-continuous.

THEOREM 4.11. Let X be a space and let $\{A_{\alpha} | \alpha \in V\}$ be a cover of X such that

- (a) each $\alpha \in V$, A_{α} is open in X or
- (b) each $\alpha \in V$, A_{α} is closed in X and the family $\{A_{\alpha} | \alpha \in V\}$ forms a neighborhood finite family.

If $f: X \rightarrow Y$ is a function such that $f|A_{\alpha}: A_{\alpha} \rightarrow Y$ is WC-continuous for each $\alpha \in V$, then f is WC-continuous.

THEOREM 4.12. If X is Urysohn and A is weakly compact relative to X, then A is closed.

Proof. Let x_0 be a point of X-A. For each $x \in A$, there exist open sets U_x and V_x containing x_0 and x, respectively, such that $\operatorname{Cl}(U_x) \cap \operatorname{Cl}(V_x) = \phi$. For each $x \in A$, we have

 $x \in \operatorname{Int}(\operatorname{Cl}(V_x)) \subset \operatorname{Cl}(V_x) \subset X - \operatorname{Cl}(U_x) \text{ and } A \subset \bigcup \left\{ \operatorname{Int}(\operatorname{Cl}(V_x)) \mid x \in A \right\}.$

Therefore, the family $\{X-\operatorname{Cl}(U_x) \mid x \in A\}$ is a cover of A by open sets of X satisfying Property (P). Since A is weakly compact relative to X, there exist a finite number of points x_1, x_2, \dots, x_n in A such that

$$A \subset \bigcup_{i=1}^{n} \operatorname{Cl}(X - \operatorname{Cl}(U_{x_i})) = X - \bigcap_{i=1}^{n} \operatorname{Int}(\operatorname{Cl}(U_{x_i})).$$

Thus, we obtain $A \cap [\cap \{\operatorname{Int}(\operatorname{Cl}(U_{x_i})) | i=1, 2, \dots, n\}] = \phi$, where $\cap \{\operatorname{Int}(\operatorname{Cl}(U_{x_i})) | i=1, 2, \dots, n\}$ is a regular open set containing x_0 . This shows that A is closed.

REMARK 4.13. The proof of Theorem 4.12 shows that A is a δ -closed set due to Veličko [13].

THEOREM 4.14. Let Y be a Urysohn space. Then, a function $f: X \rightarrow Y$ is WC-continuous if and only if $f^{-1}(K)$ is closed in X for each set K of Y weakly compact relative to Y.

Proof. This is an immediate consequence of Theorems 4.2 and 4.12.

A subset S of a space X is said to be N-closed relative to X [4] if every cover of S by regular open sets of X has a finite subcover.

THEOREM 4.15. Let X be an almost-regular space and A a subset of X. If A is weakly compact relative to X, then it is N-closed relative to X.

Proof. Let $\{V_{\alpha} | \alpha \in \mathcal{V}\}$ be a cover of A by regular open sets of X. For each $x \in A$, there exists an $\alpha(x) \in \mathcal{V}$ such that $x \in V_{\alpha(x)}$. Since X is almost-regular, there exist regular open sets $G_{\alpha(x)}$ and $W_{\alpha(x)}$ such that

$$x \in G_{\alpha(x)} \subset \operatorname{Cl}(G_{\alpha(x)}) \subset W_{\alpha(x)} \subset \operatorname{Cl}(W_{\alpha(x)}) \subset V_{\alpha(x)}.$$

The family $\{W_{\alpha(x)}|x\in A\}$ is a cover of A by open sets of X satisfying Property (P). There exists a finite subset A_0 of A such that

$$A \subset \bigcup \{\operatorname{Cl}(W_{\alpha(x)}) \mid x \in A_0\}.$$

Therefore, we have $A \subset \bigcup \{V_{\alpha(x)} \mid x \in A_0\}$. This shows that A is N-closed relative to X.

THEOREM 4.16. Let Y be an almost-regular space. Then, a function $f: X \rightarrow Y$ is WC-continuous if and only if f is H-continuous.

Proof. This is an immediate consequence of Theorem 4.15 and the fact that N-closed relative to Y implies quasi H-closed relative to Y.

THEOERM 4.17. Let Y be a regular space. Then, for a function

- $f: X \rightarrow Y$ the following are equivalent:
 - (a) WC-continuous.
 - (b) H-continuous.
 - (c) C-continuous.

Proof. Since Y is regular, Y is almost-regular and hence by Theorem 4.15 every set weakly compact relative to Y is N-closed relative to Y. Moreover, every subset of a regular space is compact if it is N-closed relative to X [4, Theorem 4.1].

THEOREM 4.18. Let Y be a compact space. Then, for a function $f: X \rightarrow Y$ the following are equivalent:

- (a) continuous.
- (b) WC-continuous.
- (c) H-continuous.
- (d) C-continuous.

Proof. By Remark 4.6, it is only necessary to show that (d) implies (a). Let F be a closed set of Y. Since Y is compact, F is compact and hence $f^{-1}(F)$ is closed in X [5, Theorem 1]. Therefore, f is continuous.

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