J. Korean Math. Soc. 24(1987), No. 1, pp. 1-9

TOPOLOGICAL COVERING MORPHISMS OF TOPOLOGICAL GROUPOIDS

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1. Introduction

The notion of topological groupoid is a generalization of that of the topological group. A useful example of topological groupoid is the fundamental groupoid πX with "lifted topology" (see [2]). An important class of topological groupoids is that of locally trivial topological groupoids. These arise in nature because of their close connection with principal bundles.

The concept of topological covering morphism between topological groupoids is a generalization of the covering map between topological spaces in the following sense: If $f: X \longrightarrow Y$ is a covering map of topological spaces, where X, Y are locally path connected and semilocally 1-connected, then the induced morphism $\pi f: \pi X \longrightarrow \pi Y$ with "lifted topology" is a topological covering morphism ([2], Theorem 4).

Some properties of the covering map can be applied to the (topological) covering morphism, but all properties cannot be applied. For example, topological covering morphism need not be open even if the covering map is open.

In section 3, we show that a topological groupoid $U \times S$ induced by a topological transformation group (U, S, ϕ) is a topological covering groupoid of the topological group S, and conversely the unit space H^0 of a topological covering groupoid H of the topological group G is a G-space (i. e. (H^0, G) forms a topological transformation group).

In section 4, under the locally trivial topological groupoid G, it is proved that the domain and range maps $d, r: G \longrightarrow G^0$ are open if G^2 is open in $G \times G$, and is proved that the Tcov-morphism $q: H \longrightarrow G$ of topological groupoids is open; using this fact, we can construct

Received December 7, 1985.
Revised October 27, 1986.

continuous lifts of morphisms (Corollary 4.4). Also we show that $q^{-1}(B)$ is a locally trivial subgroupoid of H if B is a locally trivial subgroupoid of G, where $q: H \longrightarrow G$ is a Tcov-morphism.

2. Preliminaries

A *groupoid* is a set G endowed with a product map $(x, y) \longmapsto xy: G^2 \longrightarrow G$, where G^2 is a subset of $G \times G$ called the set of composable pairs, and an inverse map $x \longmapsto x^{-1}: G \longrightarrow G$ such that the following relations are satisfied:

(a) $(x^{-1})^{-1} = x$
(b) $(x, y), (y, z) \in G^2 \Longrightarrow (xy, z), (x, yz) \in G^2$ and $(xy)z = x(yz)$
(c) $(x^{-1}, x) \in G^2$; and if $(x, y) \in G^2$, then $x^{-1}(xy) = y$
(d) $(x, x^{-1}) \in G^2$; and if $(z, x) \in G^2$, then $(zx)x^{-1} = z$.

Thus, a groupoid is a small category with inverses. Defining $d(x) = xx^{-1}$ (the domain map) and $r(x) = x^{-1}x$ (the range map), the objects or units of this category may be identified with $d(G) = r(G)$. The set $d(G)$ is said to be the *unit space* of G and will be denoted by G^0. $(x, y) \in G^2$ iff $r(x) = d(y)$; and $r(x) = x$ iff $x \in r(G)$ iff $d(x) = x$. Also the cancellation law hold; i.e. $xy = xz$ iff $y = z$. G is a group iff $G^2 = G \times G$ iff G^0 consists of exactly one element.

A groupoid G is said to be *transitive* if the map

$$(d, r): G \longrightarrow G^0 \times G^0$$

is surjective. For any $u, v \in G^0$, we will denote the set $(d, r)^{-1}(u, v)$ by $G(u, v)$. A subgroupoid N of G is said to be *normal* if, for any units u, v of G and g in $G(u, v)$, $gN(v)g^{-1} \subset N(u)$.

A *topological groupoid* consists of a groupoid G and a topology compatible with the groupoid structure:

(a) $x \longmapsto x^{-1}: G \longrightarrow G$ is continuous
(b) $(x, y) \longmapsto xy: G^2 \longrightarrow G$ is continuous,

where G^2 has the induced topology from $G \times G$. If G is a topological groupoid, then the inverse map $x \longmapsto x^{-1}: G \longrightarrow G$ is a homeomorphism and $d, r: G \longrightarrow G^0$ are continuous, where G^0 has the induced topology from G.

Let G and H be (topological) groupoids. A (continuous) map $q: H \longrightarrow G$ is called a *(TG)-morphism* if, for any $(x, y) \in H^2$, we have

$(q(x),\ q(y)) \in G^2$ and $q(x)q(y) = q(xy)$. If $u \in H^0$, then $q(u) \in G^0$ holds. Hence the restriction of TG–morphism to the unit space H^0 will be denoted by $q^0 : H^0 \longrightarrow G^0$, and said to be the *unit map*. A morphism $q : H \longrightarrow G$ of groupoids is a *covering morphism* if for each $u \in H^0$, the restriction of q mapping $St_H u \longrightarrow St_G qu$ is a bijection, where $St_H u = \bigcup_{V \in H^0} H(u, v)$.

Let $H^0 \overline{\times} G$ be given by the pull–back diagram of sets:

$$\begin{array}{ccc} H^0 \overline{\times} G & \longrightarrow & G \\ \downarrow & & \downarrow d \\ H^0 & \underset{q^0}{\longrightarrow} & G^0 \end{array}$$

Then $q : H \longrightarrow G$ is a covering morphism if and only if $(d, q) : H \longrightarrow H^0 \overline{\times} G$ is a bijection. A TG–morphism $q : H \longrightarrow G$ of topological groupoids is a *Tcov–morphism* (*topological covering morphism*) if the map $(d, q) : H \longrightarrow H^0 \overline{\times} G$ is a homeomorphism, where $H^0 \overline{\times} G$ has the induced topology from $H^0 \times G$. And H is called a *topological covering groupoid* of G.

A topological groupoid G is said to be *locally trivial* if for each $u \in G^0$ there is an open neighborhood V of u in G^0 and a continuous map $\lambda : V \longrightarrow G$ such that $\lambda(v) \in G(u, v)$ for all $v \in V$. Note that every topological group is locally trivial.

3. Topological transformation groups

Let (U, S, ϕ) be a topological transformation group by the action $\phi : U \times S \longrightarrow U$, $\phi(u, s) = us$, for $(u, s) \in U \times S$, and let G be the cartesian product set $U \times S$. Then the set G has a groupoid structure as follows; for any two elements (u, s) and (v, t) in G,

(u, s) and (v, t) are composable iff $v = us$,
$(u, s)(us, t) = (u, st)$, and
$(u, s)^{-1} = (us, s^{-1})$.

Now we have $d(u, s) = (u, s)(u, s)^{-1} = (u, e)$ and $r(u, s) = (u, s)^{-1}(u, s) = (us, e)$. Hence the map $(u, e) \longmapsto u$ identifies G^0 with U. Moreover it is easy to show that the product map $(x, y) \longmapsto xy : G^2 \longrightarrow G$ and the inverse map $x \longmapsto x^{-1} : G \longrightarrow G$ are continuous, where G has the product topology. Consequently G is a topological groupoid.

Let U be a topological space. Then the product set $U\times U$ has also a groupoid structure as follows; for any two elements (x, y) and (z, w) in $U\times U$,

$$(x, y) \text{ and } (z, w) \text{ are composable iff } y=z,$$
$$(x, y)(y, z)=(x, z), \text{ and}$$
$$(x, y)^{-1}=(y, x).$$

Furthermore the product map $(a, b)\longmapsto ab : (U\times U)^2 \longrightarrow U\times U$ and the inverse map $a\longmapsto a^{-1} : U\times U \longrightarrow U\times U$ are also continuous. Hence the product space $U\times U$ forms a topological groupoid.

LEMMA 3.1. *Let (U, S, ϕ) be a topological transformation group. Then the action ϕ induces a TG-morphism from the topological groupoid $U\times S$ to the topological groupoid $U\times U$.*

Proof. Define a map $f : U\times S \longrightarrow U\times U$ by

$$f(u, s)=(u, us),$$

where $(u, s)\in U\times S$. Then we have $f=(\pi_1, \phi)$, where $\pi_1 : U\times S\rightarrow U$ is the projection, and so the map f is continuous. Now, if $((u, s), (v, t))$ is an element of $(U\times S)^2$ then we have

$$(f(u, s), f(us, t))=((u, us), (us, (us)t))\in(U\times U)^2, \text{ and}$$
$$f((u, s)(us, t))=f(u, st)=(u, u(st))=(u, us)(us, (us)t)$$
$$=f(u, s)f(us, t)$$

Hence f is a TG-morphism.

THEOREM 3.2. *Let (U, S, ϕ) be a topological transformation group and N a normal subgroup of S. Then the topological groupoid $U\times N$ is a normal subgroupoid of the topological groupoid $U\times S$.*

Proof. For any units x, y of $U\times S$ and (u, s) in $(U\times S)(x, y)$, we get $d(u, s)=u=x$ and $r(u, s)=us=y$. If $(v, t)\in(U\times N)(y)$ then $d(v, t)=v=r(v, t)=vt=y$, and so $v=us=(us)t$. Hence we have

$$(u, s)(v, t)(u, s)^{-1}=(u, st)(us, s^{-1})$$
$$=(u, sts^{-1}).$$

Since

$$r(u, sts^{-1})=u(sts^{-1})=(us)ts^{-1}=(us)s^{-1}$$
$$=u=d(u, sts^{-1})$$

and $sts^{-1} \in N$, we get $(u, sts^{-1}) \in (U \times N)(x)$. Thus $(u, s)(U \times N)(y)(u, s)^{-1} \subset (U \times N)(x)$, and so $U \times N$ is a normal subgroupoid of $U \times S$.

THEOREM 3.3. *Let (U, S, ϕ) be a topological transformation group. Then the topological groupoid $U \times S$ is a topological covering groupoid of the topological group S.*

Proof. Let $\pi_2 : U \times S \longrightarrow S$ be the projection by $\pi_2(u, s) = s$, $(u, s) \in U \times S$. Then we can easily get that the projection π_2 is a TG–morphism. Since S is a topological group, we have

$$(U \times S)^0 \bar{\times} S = U \bar{\times} S = U \times S,$$

by the pull-back definition of $U \bar{\times} S$. Hence the map $(d, \pi_2) : U \times S \longrightarrow (U \times S)^0 \bar{\times} S$ defined by

$$(d, \pi_2)(u, s) = ((u, e), s),$$

where $(u, s) \in (U \times S)$, is a homeomorphism. Consequently π_2 is a Tcov–morphism.

Conversely, we can say that the unit space H^0 of a topological covering groupoid H of the topological group G is a G–space.

THEOREM 3.4. *Let H be a topological groupoid and G a topological group. If H is a topological covering groupoid of G then (H^0, G) forms a topological transformation group.*

Proof. Let $q : H \longrightarrow G$ be a Tcov–morphism. Then $(d, q) : H \longrightarrow H^0 \bar{\times} G$ is a homeomorphism. Since G is a topological group, we get $H^0 \bar{\times} G = H^0 \times G$. Define a map $\phi : H^0 \times G \longrightarrow H^0$ by the following diagram;

$$\begin{array}{ccc} H^0 \times G & \xrightarrow{\ \phi\ } & H^0 \\ & {\scriptstyle (d,q)^{-1}} \searrow \quad \circlearrowleft \quad \nearrow {\scriptstyle r} & \\ & H & \end{array}$$

Then the map ϕ is continuous, and we have

$$\phi(u, g) = r(\bar{g}), \quad (u, g) \in H^0 \times G,$$

where $\bar{g}$ is the lifting of g such that $q(\bar{g}) = g$ and $d(\bar{g}) = u$.
Moreover we get

$$\phi(\phi(u, g), \ h) = \phi(r(\bar{g}), \ h) = r(\overline{gh}) = \phi(u, \ gh),$$

for $g, h \in G$ and $u \in H^0$, and also we have $\phi(u, e) = u$, where e is the identity in G. Hence the pair (H^0, G) forms a topological transformation group.

4. The locally trivial case

An important class of topological groupoids is that of locally trivial topological groupoids. These arise in nature because of their close connection with principal bundles. In this section, we obtain some properites of the locally trivial topological groupoid.

THEOREM 4.1. *Let G be a locally trivial topological groupoid, and let the set G^2 of composable pairs be open in $G \times G$. Then the domain and range maps $d, r : G \longrightarrow G^0$ are open.*

Proof. We only prove that the range map $r : G \longrightarrow G^0$ is open. Similarly we can prove that the domain map $d : G \longrightarrow G^0$ is open.

Let U be any open set in G. Given any $x \in U$, there exists an open neighborhood V of $r(x)$ in G^0 and a continuous map $\lambda : V \longrightarrow G$ such that $\lambda(v) \in G(r(x), v)$, $v \in V$. Then we have either $\lambda(r(x)) \in U$ or $\lambda(r(x)) \notin U$.

As a first case, we assume that $\lambda(r(x)) \in U$. Since λ is continuous, there exists an open neighborhood W of $r(x)$ such that $\lambda(W) \subset U$. Let $V' = V \cap W$. Then the set V' is an open neighborhood of $r(x)$ and $\lambda(V') \subset U$. Hence we have $r(x) \in V' = r\lambda(V') \subset r(U)$.

As a second case, we assume that $\lambda(r(x)) \notin U$. Let $\lambda(r(x)) = y$ and $A = \{z \in G : (xy^{-1}, z) \in G^2\}$. Since the set G^2 is open in $G \times G$, the set A is also open in G. Now the map $f_{xy^{-1}} : A \longrightarrow G$ defined by

$$f_{xy^{-1}}(z) = xy^{-1}z,$$

where $z \in A$, is continuous, since the product map $(a, b) \longmapsto ab : G^2 \longrightarrow G$ is continuos. Hence there exists an open neighborhood U' of y in A such that $f_{xy^{-1}}(U') \subset U$. For such an open set U', we have $r(U') \subset r(U)$. In fact, if $a \in U'$ then $xy^{-1}a \in U$ and so $r(a) = r(xy^{-1}a) \in r(U)$. Since λ is continuous, there is an open neighborhood W' of $r(x)$ such that $\lambda(W') \subset U'$. Let $V'' = V \cap W'$. Then the set V'' is an open neighborhood of $r(x)$, and $\lambda(V'') \subset U'$. Hence we have $r(x) \in V'' = r\lambda(V'') \subset r(U') \subset r(U)$. Consequently the set $r(U)$ is open. This completes the proof.

In general, a covering map $f : X \longrightarrow Y$ of topolgical spaces is open.

But a Tcov–morphism $q : H \longrightarrow G$ of topological groupoids need not be open. In case G is locally trivial, it was proved that the unit map $q^0 : H^0 \to G^0$ is open if $q : H \longrightarrow G$ is a Tcov–morphism ([4], Proposition 11). Now we show that a Tcov–morphism $q : H \longrightarrow G$ is also open if G is locally trivial.

THEOREM 4.2. *A Tcov–morphism $q : H \longrightarrow G$ of topological groupoids is open if G is locally trivial.*

Proof. Consider the following commutative diagram;

$$\begin{array}{ccc} H & \xrightarrow{(d,q)} & H^0 \overline{\times} G \\ & {\scriptstyle q} \searrow \quad \curvearrowright \quad \swarrow {\scriptstyle \pi_2} & \\ & G & \end{array}$$

where π_2 is the projection, and (d, q) is the homeomorphism induced by the Tcov–morphism $q : H \longrightarrow G$.

Let $U_u \overline{\times} V_g$ be any open neighborhood of (u, g) in $H^0 \overline{\times} G$, and let $\pi_2(U_u \overline{\times} V_g) = \overline{V}_g$. Then it is enough to show that the set

$$\overline{V}_g = \{h \in V_g : q^{-1}(d(h)) \cap U_u \neq \phi\}$$

is open in G. Let $h \in \overline{V}_g$. Then we have $q^{-1}(d(h)) \cap U_u \neq \phi$, say $v \in q^{-1}(d(h)) \cap U_u$. Choose an open neighborhood W_v of v in H^0 contained in U_u. Since the unit map $q^0 : H^0 \longrightarrow G^0$ is open, by the Proposition 11 in [4], $q^0(W_v)$ is an open set in G^0.

Let $A = d^{-1}(q^0(W_v))$. Then the set A is open in G, since the map $d : G \longrightarrow G^0$ is continuous. Hence the set $A \cap V_g$ is an open neighborhood of h in G. Now, if $k \in A \cap V_g$, then we get $q^{-1}(d(k)) \cap W_v \neq \phi$, and so $q^{-1}(d(k)) \cap U_u \neq \phi$. Thus k is an element of the set $\overline{V}_g$. Consequently, the set $\overline{V}_g$ is open in G.

COROLLARY 4.3. *If G is a topological group then any Tcov–morphism $q : H \longrightarrow G$ is open.*

Proof. Since the topological group G is always locally trivial, the proof is clear by Theorem 4.2.

Suppose now given a diagram of morphisms of groupoids,

$$\begin{array}{ccc} & & H \\ & {\scriptstyle f^*} \nearrow & \big\downarrow {\scriptstyle q} \\ F & \xrightarrow[f]{} & G \end{array}$$

in which q is a covering morphism. Suppose further that F is transitive and that $u \in H^0$ and $v \in F^0$ satisfy $q(u) = f(v)$. Then f lifts to a morphism $f^* : F \longrightarrow H$ if and only if $f(F(v,v)) \subset q(H(u,u))$ ([1], Theorem 9.3.3).

These properties can be applied to the topological case under additional assumptions as follows; if F, G, and H are topological groupoids, f is a TG-morphism, q is a Tcov-morphism and F is locally trivial then f^* is continuous, i.e. is a TG-morphism ([4], Proposition 7).

However, if we use Theorem 4.2 then we obtain the continuity of the lifting f^*, even if F is not locally trivial.

COROLLARY 4.4. *If G is locally trivial then f^* is continuous, i.e. is a TG-morphism.*

The useful homomorphism theorem for groups (that if $f : G \longrightarrow H$ is a homomorphism then $\operatorname{Im} f$ is isomorphic to $G/\operatorname{Ker} f$) is false for groupoids, one reason being is that $\operatorname{Im} f$ need not be a subgroupoid of H. However the situation is different in the case of the inverse image.

LEMMA 4.5. *Let $q : H \longrightarrow G$ be a morphism of groupoids and B a subgroupoid of G. Then $q^{-1}(B)$ is a subgroupoid of H. Furthermore, if $q : H \longrightarrow G$ is a Tcov-morphism of topological groupoids then the restriction of q mapping $q^{-1}(B)$ into B is also a Tcov-morphism of topological groupoids.*

Proof. To prove that $q^{-1}(B)$ is a subgroupoid of H, it is enough to show that if $x, y \in q^{-1}(B)$ and $(x, y^{-1}) \in H^2$ then $xy^{-1} \in q^{-1}(B)$. Suppose that $q(x) = z \in G(a,b)$ and $q(y) = w \in G(u,v)$ for $a, b, u, v \in G^0$. Since $(x, y^{-1}) \in H^2$, we have $r(x) = d(y^{-1}) = r(y)$, and so $b = v$. Hence we get $xy^{-1} \in q^{-1}(zw^{-1}) \subset q^{-1}(B)$.

If $q : H \longrightarrow G$ is a Tcov-morphism of topological groupoids then the map $(d, p) : H \longrightarrow H^0 \overline{\times} G$ is a homeomorphism, and so the restriction $(d, p)|_q{}^{-1}(B) : q^{-1}(B) \longrightarrow q^{-1}(B)^0 \overline{\times} B$ is also a homeomor phism. Hence $q|_q{}^{-1}(B) : q^{-1}(B) \longrightarrow B$ is a Tcov-morphism.

THEOREM 4.6. *Let $q : H \longrightarrow G$ be a Tcov-morphism of topological groupoids and B a locally trivial subgroupoid of G. Then $q^{-1}(B)$ is also* *a locally trivial subgroupoid of H.*

Proof. By Lemma 4.5, it is enough to show that $q^{-1}(B)$ is locally trivial. Let $u \in q^{-1}(B)^0$ and $q^0(u) = v \in B^0$. Then there exists an open neighborhood V of v in B^0 and a continuous map $\lambda : V \longrightarrow B$ such that $\lambda(w) \in B(v, w)$, $w \in V$. Since the unit map $q^0 : H^0 \longrightarrow G^0$ is continuous, there is an open neighborhood U of u in $(H^0 \cap q^{-1}(B)) = (q^{-1}(B))^0$ such that $q^0(U) \subset V$. Define a map $\mu : U \longrightarrow q^{-1}(B)$ by the following diagram;

$$\begin{array}{ccc} U & \overset{\mu}{\longrightarrow} & q^{-1}(B) \\ (id,\ \rho\lambda q^0) \searrow & \circlearrowleft & \nearrow \rho_0(d,\ q)^{-1} \\ & U \bar{\times} B & \end{array}$$

where ρ denotes the inverse map : $x \longmapsto x^{-1}$, and $(d, q) : q^{-1}(B) \longrightarrow q^{-1}(B)^0 \bar{\times} B$ is the homeomorphism induced by the Tcov-morphism $q : q^{-1}(B) \longrightarrow$ B. Then the map μ is well-defined and continuous. Moreover, for any element u' in U, $\mu(u')$ is an element of $q^{-1}(B)(u, u')$ and also $q(\mu(u')) = \lambda(q^0(u'))$. This completes the proof.

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