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A Note on a Harmonizable Process

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§ 0. Introduction

Throughout this paper, (Ω, F, P) is the underlying probability space and $X(t, w)$, $t \in R$, is a complex valued stochastic process of the second order, where w is an element of Ω, that is

$$E\,|X(t, w)|^2 = \|X(t, w)\|^2 < \infty \text{ for every } t.$$

Harmonizable processes were first defined by Loève(4), as a special class of non stationary processes having the spectral representation

$$X(t, \omega) = \int_{-\infty}^{\infty} e^{t+\lambda}\, d\Phi(\lambda, \omega), \quad a.\,s.$$

where, $E[\Phi(\lambda, \omega)] = 0$ and $E\,|\Phi(\lambda, \omega)|^2 < \infty$ for all $\lambda \in R$ (Briefly we write $\Phi(\lambda, \omega) = \Phi(\lambda)$) such that the covariance of $\Phi(\lambda)$ and $\Phi(\lambda')$ represented by

$$E[\Phi(\lambda) \cdot \overline{\Phi(\lambda')}] = F(\lambda, \lambda')$$

is of bounded variation on the two dimensional Euclidean plane R^2 (See Prohorov, Yu. A. and Rozanov, Yu. A. (5)) and Loève has shown that in order for $X(t, \omega)$ to be harmonizable it is necessary and sufficient that the covariance function $\rho(t, t')$ has the spectral representation

$$\rho(t, t') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\lambda t - \lambda' t')}\, d^2F(\lambda, \lambda').$$

Suppose that $X(t, \omega)$ is measurable on $R \times \Omega$ and also suppose that

$$\int_a^b \|X(t, \omega)\|^2\, dt < \infty, \text{ for every finite } a < b.$$

In this case, $X(t, \omega)$ is of $L^2(a, b)$ as a function of t almost surely.

It is known(3) that for a weakly stationary process, $L^2_{u.a.p.}$ and $L^2_{s.a.p.}$ are equivalent to each other.

In this paper, we show that for a harmonizable process, $L^2_{u.a.p.}$ and $L^2_{s.a.p.}$ are also equivalent to each other. For this property it is necessary and sufficient that the covariance function $\rho(t, t')$ is $\mathfrak{U}-u.a.p.$

§ 1. Definitions

S. Bochner has developed the theory of u. a. p. functions of two variables by the following definition.

Definition 1.1. A function $\rho(t, t')$; $R^2 \to C$ which is continuous in two variables, is called uniformly almost periodic(u. a. p.) if and only if the set

$$B^2\{\varepsilon, \rho\} \equiv \{(\tau_1, \tau_2); \sup_{t,t' \in R} | \rho(t+\tau_1, t'+\tau_2) - \rho(t, t') | < \varepsilon\}$$

is relatively dense for every $\varepsilon > 0$.

As an analogue to this definition we give now the following definition.

Definition 1. 2. A function $\rho(t, t')$; $R^2 \rightarrow C$ which is continuous in two variables, is called $\mathcal{U}-u.a.p.$ with respect to t (or with respect to t') if the set

$$B\{\varepsilon, \rho\} \equiv \{\tau; \sup_{t,t' \in R} | \rho(t+\tau, t') - \rho(t, t') | < \varepsilon\}$$

$$(\text{or } \{\tau; \sup_{t,t' \in R} | \rho(t, t'+\tau) - \rho(t, t') < \varepsilon\})$$

is relatively dense for every $\varepsilon > 0$.

Remark 1. 1. *Let $\rho(t, t')$ be a covariance function of the harmonizable process $X(t, \omega)$. Then the fact that $\rho(t, t')$ is $\mathcal{U}-u.a.p.$ with respect to t', because of the fact that $\rho(t, t') = \rho(t', t)$. And the fact that $\rho(t, t')$ is $\mathcal{U}-u.a.p.$ implies $\rho(t, t')$ is u. a. p., because of the fact that, for*

$$\tau_1, \tau_2 \in B\{\varepsilon/2, \rho\},$$

$$| \rho(t+\tau_1, t'+\tau_2) - \rho(t, t') |$$

$$\leq | \rho(t+\tau_1, t'+\tau_2) - \rho(t, t'+\tau_2) | + | \rho(t, t'+\tau_2) - \rho(t, t') |$$

$$\leq \sup_{t,t' \in R} | \rho(t+\tau_1, t'+\tau_2) - \rho(t, t'+\tau_2) | + \sup_{t,t' \in R} | \rho(t, t'+\tau_2) - \rho(t, t') |$$

$$\leq \varepsilon.$$

Definition 1. 3. A function $X(t, \omega)$ which is continuous in t for almost all ω belongs to $L^2_{u.a.p.}$ if and only if the set

$$E\{\varepsilon, X\} \equiv \{\tau; \sup_{t \in R} \| X(t+\tau, \omega) - X(t, \omega) \|^2 < \varepsilon\}$$

is relatively dense for every $\varepsilon > 0$.

Definition 1. 4. $X(t, \omega) \in L^2_{s.a.p.}$ if and only the set

$$S^2\{\varepsilon, X\} \equiv \{\tau; \sup_{U \in R} \int_U^{U+1} \| X(t+\tau, \omega) - X(t, \omega) \|^2 \, dt < \varepsilon\}$$

is relatively dense for every $\varepsilon < 0$.

Definition 1. 5. For $S \in \beta(R)$, we define

$$G(S) \equiv \iint_{(\lambda \in S;\, \lambda = \lambda')} d^2F(\lambda, \lambda').$$

§ 2. Theorems

Theorem 2. 1. *Let $X(t, \omega)$ be a process of the second order and $\rho(t, t')$ be the covariance function of $X(t, \omega)$. If $X(t, \omega)$ is uniformly mean continuous then the followings are equivalent.*

(i) $X(t, \omega) \in L^2_{u.a.p.}$

(ii) $X(t, \omega) \in L^2_{s.a.p.}$

(iii) $\rho(t, t')$ *is* $\mathcal{U}-u.a.p.$

Proof. From Bochner's theorem (Besicovitch(1) p81) we have the fact that (i) implies (ii) and vice versa.

Now we show that (i) implies (iii). Let τ be an element of $E\{\varepsilon, X\}$.

Then $$\sup_{t,t' \in R} | \rho(t+\tau, t') - \rho(t, t') |$$

$$\leq \sup_{t,t' \in R} \{E|X(t+\tau) - X(t)|^2\}^{\frac{1}{2}} \cdot \{E|X(t')|^2\}^{\frac{1}{2}}.$$

since $X(t, \omega)$ is a process of the second order, there exists constant c such that the above is not greater than $c\sqrt{\varepsilon}$. Hence $B\{\varepsilon, \rho\}$ is relatively dense for every $\varepsilon>0$.

Next, we show that (iii) implies (i). Let τ be an element of $B\{\varepsilon, \rho\}$.

Then
$$\sup_{t\in R} \| X(t, \tau)-X(t) \|^2$$
$$=\sup_{t\in R} | \rho(t+\tau, t+\tau)-\rho(t+\tau, t)-\rho(t, t+\tau)+\rho(t, t) |$$
$$\leqq \sup_{t\in R} |\rho(t+\tau, t+\tau)-\rho(t+\tau, t)|+\sup_{t\in R}|\rho(t, t+\tau)-\rho(t, t)|$$
$$\leqq 2\varepsilon.$$

Hence $E\{2\varepsilon, X\}$ is relatively dence for every $\varepsilon>0$.

Theorem 2. 2. Let $X(t, \omega)$ *be harmonizable process. Then* $X(t, \omega)\in L^2_{u.a.p.}$ *and* $X(t, \omega)\in L^2_{s.a.p.}$ *are equivalent to each other.*

Proof. By Theorem 2.1., it is enough to show that $X(t, \omega)$ is uniformly mean continuous. Consider
$$E|X(t+h)-X(t)|^2$$
$$\leqq 4\left(\int_{-\infty}^{\infty}\sin^2\frac{h\lambda}{2}dG(\lambda)\right)\left(\int_{-\infty}^{\infty}dG(\lambda)\right).$$

We have, for each $A>0$,
$$\int_{-\infty}^{\infty}\sin^2\frac{h\lambda}{2}dG(\lambda)$$
$$=\int_{|\lambda|\leqq A}\sin^2\frac{h\lambda}{2}dG(\lambda)+\int_{|\lambda|>A}\sin^2\frac{h\lambda}{2}dG(\lambda)$$
$$\leqq\frac{1}{4}\int_{|\lambda|\leqq A}|h\lambda|^2dG(\lambda)+\int_{|\lambda|>A}dG(\lambda)$$
$$\leqq\frac{h^2A^2}{4}\int_{-\infty}^{\infty}dG(\lambda)+\int_{|\lambda|>A}dG(\lambda)$$
$$=\frac{A^2ch^2}{4}+\int_{|\lambda|>A}dG(\lambda), \text{ for some constant } c.$$

On the other hand we know, for every $\varepsilon>0$, there exists $A=A(\varepsilon)$ such that $\int_{|\lambda|>A}dG(\lambda)$ is less than $\varepsilon/8$. Therefore we have
$$E|X(t+h)-X(t)|^2\leqq A^2c|h|^2+\varepsilon/2.$$

Now choose $\delta=\sqrt{\varepsilon/2A^2c}>0$. Then we get, for $|h|<\delta$,
$$E|X(t+h)-X(t)|^2<\varepsilon/2+\varepsilon/2=\varepsilon.$$

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