韓國數學教育學會誌〈數學教育〉
1987, 5. Vol. XXV. No. 2

Journal of the Korea Society of Mathematical Education
May. 1987, Vol. XXV, No. 2

The Modeling of Equations of Mathematical Physics

by Dong Jin Yoo

Kyung Hee University, Seoul, Korea

1. Introduction

Partial differential equations arise in connection with various physical and geometrical problems when the functions involved depend on two or more independent variables. These variables may be the time and one or several coodinates in space. The present chapter will be devoted to some of the most important partial differential equations occuring in engineering applications.

We shall derive these equations from physical principles.

2. The modeling of two-dimensional wave equation

To derive the differential equation which governs the motion of the membrane, we consider the forces acting on a small portion of the membrane as shown in Fig.
Since the deflections of the membrane and the angles of inclination are small, the sides of the portion are approximately equal to $\triangle x$ and $\triangle y$. The tension T is the force per unit length.
Hence the forces acting on the edges of the portion are approximately $T\triangle x$ and $T\triangle y$.
Since the membrane is perfectly flexible, these forces are tangent to the membrane.

The horizontal components of the forces are obtained by multiplying the forces by the cosines of the angles of inclination.
Since these angles are small, their cosines are close to 1.
Hence the horizontal components of the forces at opposite edges are approximately equal. Therefore, the motion of the particles of the membrane in horizontal direction will be negligibly small.

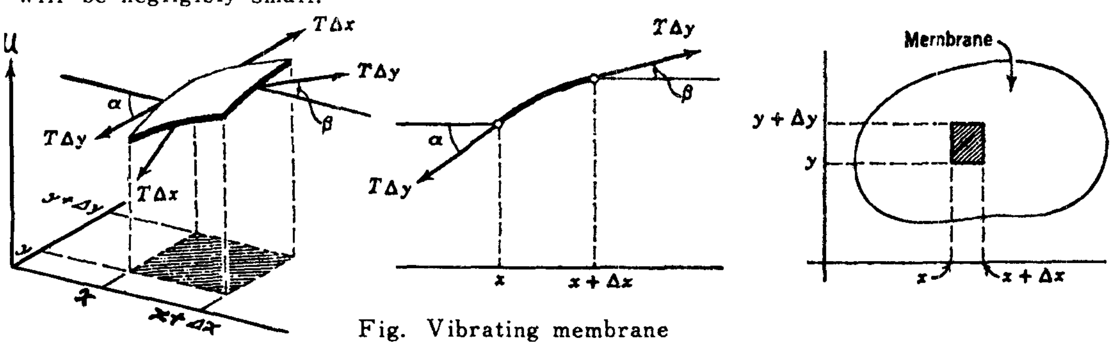

Fig. Vibrating membrane

From this we conclude that we may regard the motion of the membrane as transversal, that is, each particle moves vertically.

The vertical components of the forces along the edges parallel to the yu-plane are $T\triangle y \sin\beta$ and $-T\triangle y \sin\alpha$;

Here the minus sign appears because the force on the left edge is directed downward. Since the angles are small, we may replace their sines by their tangent. Hence the resultant of those two vertical components is

(1) $T\triangle y(\sin\beta - \sin\alpha) \simeq T\triangle y(\tan\beta - \tan\alpha)$

$= T\triangle y[U_x(x+\triangle x,\ y_1) - U_x(x,\ y_2)]$ where $y < y_1,\ y_2 < y+\triangle y$.

The resultant of vertical components of the forces acting on the other two edges of the portion is

(2) $T\triangle x[U_y(x_1,\ y+\triangle y) - U_y(x_2,\ y)]$ where $x < x_1,\ x_2 < x+\triangle x$.

y Newton's second law, the sum of the forces given by (1) and (2) is equal to the mass $\rho\triangle A$ of the portion times the acceleration $\frac{\partial^2 u}{\partial t^2}$; here ρ is the mass of the undeflected membrane per unit area and $\triangle A = \triangle x\ \triangle y$ is the area of the portion.

Thus $\rho\triangle x\triangle y \frac{\partial^2 U}{\partial t^2} = T\triangle y[U_x(x+\triangle x,\ y_1) - U_x(x,\ y_2)]$

$+ T\triangle x[U_y(x_1,\ y+\triangle y) - U_y(x_2, y)]$.

Division by $\rho\triangle x\triangle y$ yields

$$\frac{\partial^2 U}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 U}{\partial t^2} = \frac{T}{\rho}\left[\frac{U_x(x+\triangle x, y_1) - U_x(x, y_2)}{\triangle x} + \frac{U_y(x_1, y+\triangle y) - Uy(x_2, y)}{\triangle y}\right].$$

We obtain $\frac{\partial^2 U}{\partial t^2} = C^2\left(\frac{\partial^2 U}{\partial t^2} + \frac{\partial^2 U}{\partial y^2}\right)$ as $\triangle x, \triangle y \to 0$ $C^2 = \frac{T}{\rho}$.

This equation is called the two-dimensional wave equation.

$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \nabla^2 U = \triangle U$ is the Laplacian of U.

ex. Boundary and initial valve problem for the two-dimensional wave equation.

$U_{tt} - C^2\triangle U = F(x, y, t)$ $(x, y) \in \bar{R};\ t > 0$

$B(u) = 0$ on $C;\ t > 0$: (boundary condition)

$U(x, y, o) = f(x, y),\ U_t(x, y, o) = g(x, y)$ in $\bar{R}$: (initial condition).

Here F, f and g are given functions.

The solution is the form of $U(x, y, t) = \sum_{n=1}^{\infty} C_n(t)\ \varphi_n\ (x, y)\ [\varphi(x, y) = h(x)q(y)]$.

3. The modeling of the three-dimensional heat equation

We discuss the supplementary conditions that must be specified in order to determine the temperature distribution in the body.

Let Ω denote the interior of the body and $u(x, y, z, t)$ denote the temperature at the poir (x, y, z) of the body at the time t. $u(x, y, z, t)$ is C^2 with respect to the space variabl x, y, z and C^1 with respect to the time variable t.

The process of heat conduction is based on the following physical law. Let S be a smooth surface in Ω and n *denote a unit normal vector on* S.
The amount of heat energy q that crosses S to the side of the normal n in the time interval from t_1 to t_2 is given by the formula

(3.1) $$q = -\int_{t_1}^{t_2}\iint_S K(x, y, z)\,\frac{\partial u}{\partial n}\,d\sigma\,dt.$$

$\frac{\partial u}{\partial n}$ denotes the directional derivative of u in direction of the normal n at the point (x, y, z) *of* S and at the instant t.
The function $K(x, y, z)$ is positive and is called the thermal couductivity of the body at the point(x, y, z).
The heat flows in the direction of decreasing temperature.
The change in the amount of heat in the subregion A of Ω from $t=t_1$ to $t=t_2$ is given by

(3.2) $$\iiint_A C(x, y, z)\,\rho(x, y, z)\,[U(x, y, z, t_2) - U(x, y, z, t_1)]\,dxdydz.$$

C is the specific heat and ρ is the density of the body at the point(x, y, z).

According to the law of conservation of thermal energy, this change of heat in A must be equal to the amount of heat that enters into A across the boundary S in the time interval from t_1 to t_2 and this amount of heat is given by

(3.3) $$\int_{t_1}^{t_2}\iint_S K(x, y, z)\,\frac{\partial u}{\partial n}\,d\sigma dt$$

Equating the quantities (3, 2) and (3, 3) we obtain

(3.4) $$\iiint_A C(x, y, z)\,\rho(x, y, z)\,[U(x, y, z, t_2) - U(x, y, z, t_1)]\,dxdydz$$
$$= \int_{t_1}^{t_2}\iint_S K(x, y, z)\,\frac{\partial u}{\partial n}\,d\sigma dt$$

now, $U(x, y, z, t_2) - U(x, y, z, t_1) = \int_{t_1}^{t_2}\frac{\partial u}{\partial t}(x, y, z, t)\;dt$

and, since $\frac{\partial u}{\partial n} = \nabla u.\,n$, the divergence theorem applied to the vector field $V = K\nabla u$ yields

$$\iint_S K\frac{\partial u}{\partial n}\,d\sigma = \iint_S K\,\nabla u\cdot n\,d\sigma = \iint_S V\cdot n\;d\sigma = \iiint_A \nabla\cdot V\,dv$$
$$= \iiint_A \nabla\cdot(K\nabla u)\,dxdydz.$$

Hence, equation (3, 4) becomes

$$\int_{t_1}^{t_2}\iiint_A C\rho\frac{\partial u}{\partial t}\,dx\;dy\;dz\;dt = \int_{t_1}^{t_2}\iiint_A \nabla\cdot(K\nabla u)\;dx\;dy\;dz\;dt.$$

$$\therefore\ \int_{t_1}^{t_2}\iiint_A \left[C\rho\frac{\partial u}{\partial t} - \nabla\cdot(K\nabla u)\right]\,dx\;dy\;dz\;dt = 0.$$

Thus $C\rho\frac{\partial u}{\partial t} - \nabla\cdot(K\nabla u) = 0$ or $C\rho\frac{\partial u}{\partial t} - \left[\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial Z}\right)\right] = 0.$

If K, ρ and C are constant, we takes the form

$$\frac{C\rho}{K}\frac{\partial u}{\partial t}-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=0.$$

If put $k=\frac{K}{c\rho}$, $\frac{\partial u}{\partial t}=k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$.

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\nabla^2 u=\triangle u$ is Laplacian of u.

$\frac{\partial u}{\partial t}=k\nabla^2 u$ is called the three-dimensional heat equation.

ex. The boundary and initial-value problem for three-dimensional heat equation.

$U_t \ -K\ u=F(x, y, z, t)$ *in* R; $t>0$

$B(u)=H(x, y, z, t)$ *on* S; $t\geq 0$ (boundary condition)

$U(x, y, z, o)=f(x, y, z)$ *in* $\bar{R}$ (initial condition).

Here F, H and f are given functions, and $B(u)=H$ Symbolizes the boundary condition, and f describes the initial temperature.

Assume the boundary condition in above problem is homogeneous.

In order to derive a formal series solution we can represent to

$$u(x, y, z, t)=\sum_{n=1}^{\infty} C_n(t)\,\varphi_n(x, y, z)$$

where the $C_n(t)$ are to be determined.

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