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A Generalization of a Losey-Stonehewer Theorem

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— Dedicated to Professor Han Shick Park on his 60th Birthday —

1. Introduction

In 1979, G.O. Losey and S.E. Stonehewer proved the following fact:

Theorem 1. *Let G be a finite solvable group. Let U and V be p-conjugate for every prime p. Suppose that U and V have a nilpotent common normal supplement X (that is, $G=UX=VX$ and X is nilpotent and normal in G) and that one of the following conditions is satisfied:*

1) *X is abelian:*
2) *G/X is nilpotent:*
3) *the sylow p-subgroups of G have class at most 2.*

Then U and V are conjugate.

In this paper, we obtain a generalization of this theorem to the class of solvable Černikov groups.

At first, we define some terminologies. A group which is an extension of a finite direct product of quasicyclic groups by a finite group is called a Černikov group. If G is a Černikov group, G^0 will denote the unique subgroup of G minimal with respect to having finite index in G. The subgroup G^0 is the unique maximal radicable subgroup of G; that is, if $x\in G^0$ and $n\in Z^+$, then there exists $y\in G^0$ such that $y^n=x$. It is well-known that G^0 is the direct sum of finitely many quasicyclic groups (Prüfer p-groups) for (possibly) different primes p.

If G is a Černikov group and G^0 is the minimal subgroup of finite index in G, then we define the rank of G, $r(G)=\sum_{p\in\pi(G^0)} rank\ (\gamma_p(G^0))$. where $\gamma_p(G^0)$ is the p-primary component of G^0 and $\pi(G^0)$ is the set of primes which divide the order of elements of G^0. And we define $i(G)=|G/G^0|$. The pair $(r(G),\ i(G))$ will be called the size of G, denoted by $|G|$, We well-order the sizes of Černikov groups lexicographically. Thus if H is another Černikov group then we define $|G|<|H|$ if either $r(G)<r(H)$ or $r(G)=r(H)$ and $i(G)<i(H)$.

Two subgroups H and K are said to be p-conjugate in G if the sylow p-subgroups of H and the sylow p-subgroups of K are conjugate.

2. Main Result

We prove the following theorem.

Theorem 2. *Let G be a solvable Černikov group. Let U and V be p-conjugate for every prime p.*

Suppose that U and V have a locally nilpotent common normal supplement X (that is, $G=UX=VX$ and X is locally nilpotent and normal in G) and that one of the following conditions is satisfied:

1) *X is abelian;*
2) *G/X is locally nilpotent;*
3) *the sylow p-subgroups of G have class at most 2.*

Then U and V are conjugate.

Proof.

claim 1. $G^0=U^0X^0=V^0X^0$

For, $|G: U^0X^0|=|G: UX^0||UX^0: U^0X^0|<\infty$. Similarly, $|G: V^0X^0|<\infty$.
So, $G^0=U^0X^0=V^0X^0$.

claim 2. *If X is finite, Theorem 2 holds.* For, by claim 1, $G^0=U^0=V^0$.
So, $G/U^0=(U/U^0)(XU^0/U^0)=(V/V^0)(XV^0/V^0)$
Now by Theorem 1, U and V are conjugate.

claim 3. *If X is infinite abelian, Theorem 2 holds.*

proof. We know that an abelian group satisfies the minimal condition if and only if it is direct sum of finitely many quasicyclic groups and cyclic groups of prime power order. So if F is a finite subset of X, then we can find a finite characteristic subgroup W of X that contains F.

Now let $\pi(G)=\{p_1, \cdots, p_k\}$

$U_{p_i}\in Syl_{p_i}(U)$, $i=1, \cdots, k$

$V_{p_i}\in Syl_{p_i}(V)$, $i=1, \cdots, k$.

We can find $x_1, y_1\in X$, $u_1\in U$, $v_1\in V$ such that $U_{p_i}=V_{p_i}^{v_ix_i}$, $i=1, \cdots, k$ and $V_{p_i}=U_{p_i}^{u_ix_i}$, $i=1, \cdots, k$.
Now we can find a characteristic subgroup W of X such that

$$\{x_1, \cdots, x_k, y_1, \cdots, y_k\}\subseteq W$$

Then $W\trianglelefteq G$.
Consider $G^*=UW=VW$. It is clear that $UW=VW$,
W is finite, abelian, normal subgroup of G^*, and U and V are p_i-conjugate in G^*, $i=1, \cdots, k$.
By claim 2, U and V are conjugate in G^*, and hence in G.

claim 4. *If G/X is locally nilpotent, our Theorem holds.*

proof. Clearly we may assume that $G^0\leq X$ and X is infinite. Hence G/X is finite nilpotent and $G^0=X^0$.

(A) At first, we assume that U and V are finite.

We can find a finite subgroup F of X such that $X=G^0F$. Since X is normal,

$$F^{<U,V>}=<f^\alpha : f\in F,\ \alpha\in<U, V>>$$

is a finite subgroup of X.

Now if U_{p_i} and V_{p_i} are sylow p_i-subgroups of U and V respectively, we can find $x_i, y_i\in X$, $u_i\in U$, $v_i\in V$ such that $U_{p_i}=V_{p_i}^{v_ix_i}$, $V_{p_i}=U_{p_i}^{u_iy_i}$.
Since $U\cap X$ and $V\cap X$ are finite, we can find n such that

1) $x_i, y_i \in (G^0)_n F$ for all i,

2) $F^{<U,V>} \leq (G^0)_n F$,

3) $X \cap U \leq (G^0)_n F$, $X \cap V \leq (G^0)_n F$.

If we denote $(G^0)_n F$ by X^*, it is clear that X^* is finite nilpotent and X^* is normal in $G^* = X^* U = X^* V$.

By the construction of X^*, U and V are p_i-conjugate in G^* for all i and G^*/X^* is nilpotent. By the Losey-Stonehewer's result, U and V are conjugate in G^*, and hence in G.

(B) Now we assume that U and V are infinite.

Suppose that the Theorem is false. So there would exist a counterexample G of minimal size.

1) *U and V contain no normal subgroups of G which are infinite.*

proof. Suppose there is an infinite subgroup $N \trianglelefteq G$ with $N \leq U$. Since U and V are locally conjugate, $N \leq V$ also.

Let $\alpha : G \longrightarrow G/N$ be the natural homomorphism.

Then X^α is a locally nilpotent common normal supplement of U^α and V^α in $G^\alpha = G/N$, U^α and V^α are conjugate for every prime p and G^α/X^α is nilpotent. Therefore, because $|G^\alpha| < |G|$, it follows that $U^\alpha = U/N$ and $V^\alpha = V/N$ are conjugate in $G^\alpha = G/N$ and hence U and V are conjugate in G, giving a contradiction.

2) *X is a p-group, for some prime p.*

proof. Otherwise, let p be a prime in $\pi(X)$ such that p-primary part of X is infinite. Since X is locally nilpotent, $X = A \times B$ where $A \neq 1 \neq B$, A is a p-group and B is a p'-group. Then A and B are normal in G. Let

$$\alpha : G \longrightarrow G/A$$

be the natural homomorphism. As in 1), $U^\alpha = UA/A$ and $V^\alpha = VA/A$ are conjugate in $G^\alpha = G/A$. Thus there is an element g in G such that

$$UA = (VA)^g = V^g A.$$

Replacing V^g by V, we may assume, without loss of generality, that

$$UA = VA = H.$$

Now we have

$$G/B = HB/B \xrightarrow{\beta} H/(H \cap B)$$

where β is the natural isomorphism. Again by choice of G, UB/B and VB/B are conjugate in G/B and hence their image $(H \cap UB)/(H \cap B)$ and $(H \cap VB)/(H \cap B)$, under β, are conjugate in $H/(H \cap B)$.

Therefore $H \cap UB$ and $H \cap VB$ are conjugate in H.

However, if $h \in H \cap UB = UA \cap UB$, then

$$h = u_1 a = u_2 b, \quad u_1, u_2 \in U, \quad a \in A, \quad b \in B,$$

and so

$$ab^{-1} = u_1^{-1} u_2 \in U.$$

Using the fact that a and b are commuting elements of coprime order, it follows that $a, b \in U$. Hence $h = u_1 a \in U$ and so

$$U \leq UA \cap UB = H \cap UB \leq U$$

i.e. $U = H \cap UB$. Similarly $V = H \cap VB$ and U, V are conjugate in H, a contradiction.

3) $U\cap X=V\cap X$ is finite.

proof. Since U and V are p-conjugate, we may assume that U and V have a common sylow p-subgroup P.

Let $H=<U, V>$ and $X_1=X\cap U$. By hypothesis H/X_1 is finite nilpotent and X_1 is a locally nilpotent common normal supplement of U and V in H. Also U and V are p-conjugate in H. Moreover, for $q\neq p$, it follows from 2) that a sylow q-group of U (or of V) is also a sylow q-subgroup of G and therefore of H. Hence U and V are p-conjugate in H for every prime p. If $H<G$, then, by choice of G, U and V are conjugate in H, contradicting our initial assumption. So we must gave

$$G=H=<U, V>.$$

Now $X\cap U\trianglelefteq U$ and thus $X\cap U\leq P$. Therefore $X\cap P=X\cap U\trianglelefteq U$ and similarly $X\cap P=X\cap V\trianglelefteq V$. Then $X\cap P\trianglelefteq<U, V>=G$ and by 1) $X\cap P=X\cap U=X\cap V$ is finite.

4) Since we assume that U and V are infinite and $G^0\leq X$, 3) is a contradition. Hence no counterexample to our Theorem exists.

claim 5. *If the sylow p-subgroup of G have class at most 2, then our Theorem holds.*

proof. Also in this case, we may assume that $G^0\leq X$ and X is infinite. Hence G/X is finite and $G^0=X^0$.

A) Assume that U and V are finite. We know that we can find G^*, X^* so that

$$G^*=X^*U=X^*V,$$
$$X^*\trianglelefteq G^*$$

U and V are p_i-conjugate in G^*, and G^* is finite. Since the property(*) that the sylow sub groups have class at most 2 is subgroup closed, by Losey-Stonehewer's result, U and V are conjugate.

B) Assume that U and V are infinite. Since the property(*) is also quotient closed, by the same argument used in the proof of claim 4, we can show that no counterexample to our theorem exists.

3. Remarks

In fact, we can prove the following theorem.

Theorem 3. *Let G be a locally finite-solvable CC-group or a locally finite-solvable group with min-p, for all primes p. Let U and V be p-conjugate for every prime p. Suppose that U and V have a locally nilpotent common normal supplement X and that one of the following conditions is satisfied.*

1) *X is abelian;*

2) *G/X is locally nilpotent;*

3) *the sylow p-subgroups of G have class at most 2. Then U and V are locally conjugate.*

In theorem 3, U and V need not be conjugate. Consider the following example.

Let $\{p_i\}$ be an infinite set of different primes.

Let $H_i=<a_i, b_i \mid a_i^{p_i^2}=1=b_i,\ b_i^{-1}a_ib_i=a_i^{p_i+1}>$, $X_i=<a_i>$, $U_i=<b_i>$, and $V_i=<b_i^{a_i}>$.

Now let $G=\overset{\infty}{\underset{i=1}{Dr}}\ H_i$, $X=\overset{\infty}{\underset{i=1}{Dr}}\ X_i$, and $V=\overset{\infty}{\underset{i=1}{Dr}}\ V_i$.

Then G is nilpotent of class 2. G is also a countable $\mathscr{X}$-group (in fact, a countable $\sum$-group) and a locally finite-solvable FC-group.

Note that

$$G=UX=VX,\quad X\trianglelefteq G, X\text{: abelian}$$

and U and V are p_i-conjugate for all p_i.

Although U and V are locally conjugate, U and V are not conjugate.

Reference

G.O. Losey, S.E. Stonehewer, *Local Conjugacy In Finite Soluble Groups*, Quart. J. Math. Oxford (2), 30(1979), 183~190.