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Posets Related to Some Association Schemes

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— Dedicated to Professor Han Shick Park on his 60th birthday —

Abstract

A geometric interpretation of t-designs, in the context of posets, is given in the association cheme, not isomorphic with the Hamming scheme $H(2,4)$, but having the same parameters as $H(2,4)$. It is also shown that there is no similar poset which characterizes the designs for any f the three exceptional association schemes, not isomorphic with the Johnson scheme $J(8,2)$, but aving the same parameters as $J(8,2)$.

1. Introduction

In a symmetric association scheme, an algebraic definition of t-design has been introduced by Delsarte [6]. It is known that the combinatorial meaning of t-designs in $H(n,q)$ coincides with the oncept of the orthogonal array of strength t with n-rows over F. In the case of $J(v,k)$, t-designs re nothing but the classical ordinary t-(v,k,λ) designs [6]. Geometric interpretations of t-designs ave been discussed by several authors such as Delsarte [7]. Stanton [13], Bannai and Ito [2], Munmasa [10], etc. In particular, for most of the known (P and Q)-polynomial association schemes it known that there is a natural geometric interpretation of t-designs in the context of posets (semi-ttices), and hence there is a geometric definition of t-designs which coincides with the algebraic ne [2].

In this paper we want to find the most natural poset, which characterizes t-designs in the excep-onal cases of Hamming and Johnson schemes. In Section 3 we give a regular poset which char-cterizes t-designs in the Shrikhande graph G_M, the exceptional case of Hamming scheme $H(2,4)$, nd show that the poset is unique under a certain regularity condition. In Section 4 we show that ere is no such regular poset which characterizes the designs in any of the exceptional cases of hnson scheme $J(8,2)$. Before we discuss our results, in Section 2 we define our terms and give me background information to make this work self-contained. We refer to Bannai and Ito [1, 2], Delsarte [6] for further details.

2. Definitions and Preliminaries

We use $[d]$ for the set $\{0,1,\cdots,d\}$, I for the unit matrix, and J for the all-one matrix.

Definition 2.1. A *symmetric association scheme of class d*, for short a *scheme*, is a configuration $\mathscr{X}=(X, \{R_i\}_{i\in[d]})$ consisting of a finite set X and symmetric relations $R_0, R_1, \cdots, R_d$ on X where

1. $R_0=\{(x,x) : x\in X\}$ is the identity relation,
2. $\{R_i\}_{i\in[d]}$ is a partition of $X\times X$,
3. For any h, i, $j\in[d]$ and any x, $y\in X$ with $(x,y)\in R_h$, the number $z\in X$ where $(x,z)\in R_i$ and $(z,y)\in R_j$ depends only on h, i, and j. This number is denoted by $p_i{}^h{}_j$.

We call $p_i{}^h{}_j$ the ***intersection numbers*** (***parameters***) and R_i the ***i-th associate class of*** $\mathscr{X}$. Let the adjacency matrices $I=A_0, A_1, \cdots, A_d$ of $\mathscr{X}$ have rows and columns labelled by X, and for each i the (x,y)-entry of the i-th adjacency matrix $(A_i)_{xy}=1$ or 0 according as x and y are i-th associate or not. By 3 of Definition 2.1 these (0,1)-matrices satisfy

$$A_iA_j=\sum_{h\in[d]}p_i{}^h{}_jA_h \quad (i,\ j\in[d]),$$

so they span an algebra (semi-simple algebra) $\mathscr{A}(\mathscr{X})$ over R, called the ***adjacency algebra*** or ***Bose Mesner algebra***. Let $E_0, E_1, \cdots, E_d$ be the primitive idempotents of $\mathscr{A}(\mathscr{X})$, ordered so $E_0=|X|^{-1}J$. Let P and Q be the degree $d+1$ matrices whose (i,j)-th entries $p_j(i)$ and $q_j(i)$ are defined by

$$A_j=\sum_{i\in[d]}p_j(i)E_i$$

and

$$E_j=|X|^{-1}\sum_{i\in[d]}q_j(i)A_i$$

We call P and Q the ***first and second eigenmatrix***, respectively. Let the i-th intersection matri B_i of $\mathscr{X}$ be the matrix of degree $d+1$ where (j,h)-th entry is the parameter $p_i{}^h{}_j$. We note that

$$B_iB_j=\sum_{h\in[d]}p_i{}^h{}_jB_h$$

and the algebra spanned by $B_0, B_1, \ldots, B_d$ is isomorphic to the adjacency algebra $\mathscr{A}(\mathscr{X})$ by the co respondence of A_i to B_i. In particular, B_i and A_i have the same minimal polynomial.

A ***graph*** G is a pair (V,E) consisting of a set V of v vertices and a set E of unordered pai of vertices, called edges. The ***adjacency matrix of a graph*** G is the symmetric matrix A who rows and columns are labelled by the vertices, and whose (u,v)-entry is 1 if $\{u,v\}$ is an edge, a 0 otherwise. The ***distance*** $d(u,v)$ between u and v is the length of the shortest path joining u a v. The ***diameter*** d of G is the maximum distance of two vertices. For u and v in V, and f $d(u,v)=h$, we let

$$p_i{}^h{}_j(u,v)=|\{w\in V : d(u,w)=i,\ d(v,w)=j\}|.$$

If $p_i{}^h{}_j(u,v)$ is constant for all u,v with $d(u,v)=h$, then we write it as $p_i{}^h{}_j$. We call $G=(V,E)$ ***distance regular graph*** if $\{p_i{}^h{}_j\}_{i,j,h}$ are constants. With a distance regular graph of diameter we associate the association scheme$(V, \{R_i\}_{i\in[d]})$ where $(u,v)\in R_i$ if and only if $d(u,v)=i$.

In this discussion we are going to deal with a special class of asscciation schemes, name (P and Q)-polynomial association schemes.

A symmetric association scheme $\mathscr{X}=(X, \{R_i\}_{i\in[d]})$ is called a ***P-polynomial association sche*** with respect to the ordering $R_0, R_1, \cdots, R_d$ if there exist some complex coefficient polynomials $v_i($ of degree i, $i\in[d]$, such that $A_i=v_i(A_1)$ with respect to the ordinary matrix multiplicati (Then the eigenmatrix P satisfies $p_i(j)=v_i(p_1(j))$ for the same polynomial $v_i(x)$, and vice vers particular, the combinatorial concept of P-polynomial association schemes is equivalent to that listance regular graphs. That is, a P-polynomial association scheme $(X, \{R_i\}_{i\in[d]})$ is a symmet- association scheme such that a pair (x, y) of element x, $y\in X$ is in the relation R_i if and r if $d(x,y)=i$ in the undirected graph (X, R_1), with vertex and edge sets being X and R_1, ectively.

et $\mathscr{X}=(X, \{R_i\}_{i\in[d]})$ be a symmetric asscciation scheme and let E_i $(i=1, 2, \cdots, d)$ be the primitive ıpotents of the adjacency algebra $\mathscr{A}(\mathscr{X})$. $\mathscr{X}$ is called a *Q-polynomial association scheme* with ect to the ordering $E_0, E_1, \cdots, E_d$, if there exist some complex coefficient polynomials $v_i{}^*(x)$ ı that $|X|E_i=v_i{}^*(|X|E_1)$ and $v_i{}^*(x)$ has degree i, where the multiplication of $\mathscr{A}(\mathscr{X})$ is under entrywise (Hadamard) product. Equivalently, the second eigenmatrix Q satisfies, for $i, j\in[d]$, $)=v_i{}^*(q_1(j))$ for the same polynomials.

e will need two important classes of (P and Q)-polynomial association schemes, namely Ham- ; and Johnson schemes.

t F be a set of cardinality $q(q\geq 2)$ and $X=F^n$, the n-th Cartesian power of F. The *Hamming nce* between two points $x=(x_1, x_2, \cdots, x_n)$ and $y=(y_1, y_2, \cdots, y_n)$ of X is

$$d(x, y)=|\{i: x_i\neq y_i,\ i=1, 2, \cdots, n\}|.$$

R_i be the i-th distance relation on X, i. e.

$$R_i=\{(x, y)\in X\times X: d(x, y)=i\}$$

$\mathscr{X}=(X, \{R_i\}_{i\in[n]})$ is a symmetric((P and Q)-polynomial) association scheme and is called the *ming scheme* $H(n, q)$ of length n over F. We call the associated distance regular graph $R_1)$ the *Hamming graph* of the scheme $H(n, q)$ and denote it also by $H(n, q)$.

V be a set of cardinality v and $X=\mathscr{P}_k(V)$, the family of all k-element subsets *of* $\leq\frac{1}{2}v\Big)$. We define the distance between x and y of X by

$$d(x, y)=k-|x\cap y|.$$

R_i be the i-th distance relation on X, i. e.,

$$R_i=\{(x, y)\in X\times X: |x\cap y|=k-i\}.$$

$\mathscr{X}=(X, \{R_i\}_{i\in[k]})$ is a symmetric ((P and Q)-polynomial) association scheme and called the *on scheme* $J(v, k)$. The graph (X, R_1) is called the Johnson graph and also denoted by).

we define the *t-design* in a Q-polynomial association scheme $\mathscr{X}=(X, \{R_i\}_{i\in[d]})$ of class d. non-empty subset Y of X, the $(d+1)$-tuple $\underset{\sim}{a}=(a_0, a_1, \cdots, a_d)$ is said to be the *inner distri-* of Y with respect to $\{R_i\}_{i\in[d]}$ if a_i is given by

$$a_i=|Y|^{-1}|(Y\times Y)\cap R_i|, \text{ for } i=0, 1, \cdots, d.$$

the dual $\underset{\sim}{b}=(b_0, b_1, \cdots, b_d)$ of $\underset{\sim}{a}$ by $\underset{\sim}{b}=|Y|^{-1}\underset{\sim}{a}Q$ where Q is the second eigenmatrix of the e. Obviously, $a_0=b_0=1$, $b_i=a_0q_i(0)+a_1q_i(1)+\cdots+a_dq_i(d)$.

nition 2. 2. A non-empty subset Y of X is called a *t-design* in a Q-polynomial association $\mathscr{X}=(X, \{R_i\}_{i\in[d]})$ with respect to $\{E_0, E_1, \cdots, E_d\}$ if $b_1=b_2=\cdots=b_t=0$.

ıe Johnson scheme $J(v, k)$, a subset Y of X is a t-design if and only if Y forms an ordinary combinatorial t-(v, k, λ) design in the v-element set V, i.e., each t-element subset of V is con tained in a constant number λ of elements of Y. In the Hamming scheme $H(n, q)$, Y is a t-design if and only if Y forms an orthogonal array of strength t with n rows over F(see [6, p. 43, 51]).

Through out this paper, a ranked poset $(P, \leq)$ (simply, poset P) is a finite graded poset [4 $P = X_0 \cup X_1 \cup \cdots \cup X_d$, where $\mathrm{rank}(X_i) = i$, $X_0 = \{0\}$ is the unique minimal element of P, and $X_d =$ is the set of maximal elements of P. It is known that most of the known (P and Q)-polynomial association schemes have certain posets which characterize their t-designs in the following sense: subset Y of X is a t-design in a Q-polynomial association scheme $\mathscr{X} = (X, \{R_i\}_{i \in [d]})$ if and only $\lambda_i(z) = |\{y \in Y : z \leq y\}|$ is constant for all $z \in X_i$ and all $i = 0, 1, \cdots, t$, in the poset P *attached* $\mathscr{X}$. For instance, we can attach certain posets to the Johnson and Hamming schemes as follows:

1. For $J(v, d)$, let $P = X_0 \cup X_1 \cup \cdots \cup X_d$ be defined by $X_i = \mathscr{P}_i(V)$, the set of all i-eleme subsets of V, and $x \leq y$ in P if and only if $x \subseteq y$ in V.
2. For $H(d, q)$, let $P = X_0 \cup X_1 \cup \cdots \cup X_d$ be defined by $X_i = \{f \in F^J : |J| = i, J \subseteq [d]\}$, whe F^J is the set of all functions from J to F, and $f \leq g$ for $f \in F^J$, $g \in F^K$ with $J, K \subseteq [d]$ and only if $J \subseteq K$ and $g|_J = f$.

Then these posets are ones which characterize t-designs for the Johnson and Hamming schemes the above sense. The construction of right posets has been carried out in each individual class association schemes by several authors (for instance, see [2], [7], [10], [13]).

3. Posets Attached to G_M

By Egawa [9] it is known that the Hamming scheme $H(n, q)$ is uniquely determined by the parameters if $q \neq 4$. However, if $q = 4$ there are $\left[\frac{1}{2}n\right]$ exceptional graphs of $H(n, 4)$, not isomorphic with $H(n, 4)$, but having the same parameters as $H(n, 4)$. Inparticular, if $n = 2$ then there is one exceptional graph, the Shrikhande graph G_M which is depicted in Figure 1. The vertices labelled with the same index are identified. The aim of this section is to find a natural poset which characterizes the designs in G_M, We start with classifying all the designs in G_M. We use the convention that the set of vertices of G_M.

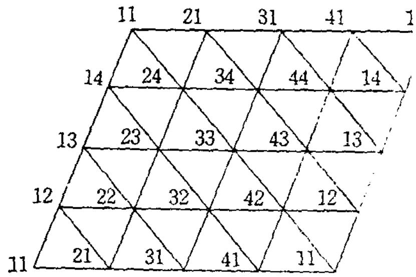

Figure 1. G_M

Lemma 3.1. *If Y is an 1-design in G_M, then the size of Y is an element of* $\{4, 6, 8, 10, 12,$

Proof. Let $Y \subseteq X$ be an 1-design of size y in G_M, and let $\underline{a} = (a_0, a_1, a_2)$ and $\underline{b} = (b_0, b_1, b_2$ the inner and dual distribution of Y. From the definition of t-design with $t = 1$, with the se eigenmatrix

$$Q = \begin{bmatrix} 1 & 6 & 9 \\ 1 & 2 & -3 \\ 1 & -2 & 1 \end{bmatrix},$$

we have a system of equations

$$y^{-1}(1 + a_1 + a_2) = 1,$$

$$y^{-1}(6+2a_1-2a_2)=0,$$

ler the condition that $a_i=y^{-1}y_i (i=1,2)$, where $y_i=|(Y\times Y)\cap R_i|$ with y and y_i integers. ving the equations we have seven triples $(y;\ a_1,a_2)$ as solutions, namely, $(4;\ 0,3)$, $(6;\ 1,4)$, ;2,5), (10; 3,6), (12; 4,7), (14; 5,8), (16; 6,9). Among these the case (14; 5,8) can not ur because $y_1\geq 72$ for any 14 vertex subset of X in G_M. Hence the size of any 1-design in G_M ıst be one of 4,6,8,10,12, or 16.

For our convenience, we denote the collection of all 1-designs of type $(y;\ a_1,a_2)$ by $\mathcal{F}_i$ where index $i=a_1$. We can easily find all 1-designs of each type. For instance and later use, by colting all 4-vertex subsets in which no two vertices are adjacent, we get all 1-designs of size 4. fact $|\mathcal{F}_0|=16$ and the members of $\mathcal{F}_0$ are

$$\{11,22,33,44\},\ \{12,23,34,41\},\ \{13,24,31,42\},\ \{14,21,32,43\},$$
$$\{11,13,32,34\},\ \{21,23,42,44\},\ \{31,33,12,14\},\ \{41,43,22,24\},$$
$$\{11,23,31,43\},\ \{12,24,32,44\},\ \{13,21,33,41\},\ \{14,22,34,42\},$$
$$\{11,13,31,33\},\ \{12,14,32,34\},\ \{21,23,41,43\},\ \{22,24,42,44\}.$$

nilarly, by collecting all 6-vertex subsets in which each vertex has only one adjacent vertex, can find all members of $\mathcal{F}_1$. Then $\mathcal{F}_i$, for $i=2,3,4$, can also be found by taking union of two joint 1-designs of $\mathcal{F}_0\cup\mathcal{F}_1$. We also note that if Y is an 1-design of G_M then the complement Y in X is also 1-design of G_M. We summerize this in the following lemma.

Lemma 3.2. $|\mathcal{F}_0|=|\mathcal{F}_4|=16$, $|\mathcal{F}_1|=|\mathcal{F}_3|=32$, $|\mathcal{F}_2|=18$.

Let $P=X_2\cup X_1\cup X_0$ be a rank two poset with $X_2=X$, the vertex set of G_M, and $X_0=\{0\}$, the nimal element. For each $x\in X$, let $B(x)=\{s\in X_1 : s\leq x\}$ and $\beta_x=|B(x)|$. Also for each $s\in X_1$, $C(s)=\{x\in X : x\geq s\}$ and $\gamma_s=|C(s)|$. Suppose β_x(resp. γ_s) does not depend on the choice of x sp. s) then we write the constant β (resp. γ). For $Y\subseteq X$ and $s\in X_1$, $\lambda_Y(s)$ denotes the size of $\{y\in Y : y\geq s\}$. Suppose $\lambda_Y(s)$ is constant for all $s\in X_1$, we write it as λ_Y and we say that exists'. For a family $\mathcal{F}$ of subsets of X, let λ_Y exist for all $Y\in\mathcal{F}$, and let λ_Y be the same all $Y\in\mathcal{F}$, then we denote this constant by $\lambda_{\mathcal{F}}$ and say that '$\lambda_{\mathcal{F}}$ exists'. For each $x\in X$, let $(x)=\{Y\in\mathcal{F} : x\in Y\}$ and denote $|\mathcal{F}(x)|$ by ϕ_x. If ϕ_x is constant for all $x\in X$ with respect to family $\mathcal{F}$, then we denote this constant by $\phi_{\mathcal{F}}$, and say that '$\phi_{\mathcal{F}}$ exists'.

Definition 3.3. P is said to be an *'attachable' poset* to G_M if P satisfies the following statent: Y is a 1-design in G_M if and only if λ_Y exists.

Lemma 3.4. *Suppose there is a poset attachable to G_M, then*

1. *γ exists,*
2. *$\lambda_Y=\lambda_Z$ if and only if $\sum_{y\in Y}\beta_y=\sum_{z\in Z}\beta_z$,*
3. *If $\mathcal{F}$ is a family of subsets of X and $\lambda_{\mathcal{F}}$ and $\phi_{\mathcal{F}}$ exist, then $|\mathcal{F}|\lambda_{\mathcal{F}}=\gamma\phi_{\mathcal{F}}$.*

Proof. Since X itself is a 2-design in G_M so 1. is obvious. 2. and 3. are also immediate by basic counting arguments.

Remark. From the preceding paragraph of Lemma 3.2 we can easily see that $\phi_{\mathcal{F}_i}$ exists for each the 1-design families in G_M. In fact, $\phi_{\mathcal{F}_0}=4$, $\phi_{\mathcal{F}_1}=12$, $\phi_{\mathcal{F}_2}=9$, $\phi_{\mathcal{F}_3}=20$, and $\phi_{\mathcal{F}_4}=12$.

Lemma 3.5. *Let $\mathscr{F}_i$ be the family of* 1*-designs in G_M as we defined. Suppose there is an attachable poset P for which $\lambda_{\mathscr{F}_i}$ exists for each i. Then $\gamma=8$ and $\lambda_{\mathscr{F}_i}=i+2$ for $i=0, 1, 2, 3, 4$.*

Proof. By 3 of Lemma 3.4., $\gamma=|\mathscr{F}_i|\lambda_{\mathscr{F}_i}/\phi_{\mathscr{F}_i}$, where all variables are positive integers with $\gamma\leq 16$. Therefore, from the above remark,

$$\gamma=4\lambda_{\mathscr{F}_0}=32\lambda_{\mathscr{F}_1}/12=2\lambda_{\mathscr{F}_2}=\cdots.$$

implies $\gamma=8$ and $\lambda_{\mathscr{F}_i}=i+2$.

The existence of such poset is guaranteed by the following theorem.

Theorem 3.6. *There exists a non-trivial poset P (for which $\lambda_{\mathscr{F}_0}=2$) attachable to G_M.*

Proof. We construct a poset $P=X\cup X_1\cup X_0$ for which $\lambda_{\mathscr{F}_0}=2$, then we show this poset is th one which is attachable to G_M. Suppose there is a poset P which satisfies $\lambda_Y=2$ for all $Y\in\mathscr{F}_0$ Then for each member s of X_1 and for every 1-design $Y\in\mathscr{F}_0$, $|C(s)\cap Y|=2$, in particular with $Y=\{11, 22, 33, 44\}$, $|C(s)\cap\{11, 22, 33, 44\}|=2$. Without loss of generality, we assume tha $C(s)\cap\{11, 22, 33, 44\}=\{11, 22\}$. Then from the design $\{22, 24, 42, 44\}$ one of 24 and 42 (but nc both) must belong to $C(s)$ since $|C(s)\cap\{22, 24, 42, 44\}|=2$. Suppose we assume the case tha $24\in C(s)$, then it follows $21\in C(s)$ and $23\in C(s)$ from the design $\{21, 23, 42, 44\}$. Moreover, 3 and 43 can not belong to $C(s)$ from the design $\{11, 23, 31, 43\}$, $13\in C(s)$ from $\{11, 13, 31, 33\}$, an 12 and 14 belong to $C(s)$ from $\{31, 33, 12, 14\}$. After all, $C(s)=\{11, 12, 13, 14, 21, 22, 23, 24\}$. On th other hand, suppose we assume $42\in C(s)$ instead of 24, then 14 and 34 can not belong to $C(s)$ froı the design $\{14, 22, 34, 42\}$, and 12 and 32 belong to $C(s)$ from $\{12, 24, 32, 44\}$. Moreover, $31\in C($ from $\{12, 14, 31, 33\}$, and thus, 13, 23, and 43 can not belong to $C(s)$ from the designs $\{13, 24, 31, 4$ and $\{11, 23, 31, 43\}$. Hence this time we have $C(s)=\{11, 12, 21, 22, 31, 32, 41, 42\}$. Similarly, we start with $C(s)\cap\{11, 22, 33, 44\}=\{11, 33\}$, then $C(s)$ is either $\{11, 12, 21, 24, 33, 34, 42, 43\}$ ($\{11, 14, 23, 24, 32, 33, 41, 42\}$. In this manner, we are able to find all possible candidates for tl members of X_1. In fact, there are 12 candidates (name them $s_1, s_2, \cdots, s_{12}$). Their upper bounds ar

$$\begin{array}{ll}
\{11, 12, 13, 14, 21, 22, 23, 24\}, & \{31, 32, 33, 34, 41, 42, 43, 44\}, \\
\{21, 22, 23, 24, 31, 32, 33, 34\}, & \{11, 12, 13, 14, 41, 42, 43, 44\}, \\
\{11, 12, 21, 24, 33, 34, 42, 43\}, & \{13, 14, 22, 23, 31, 32, 41, 44\}, \\
\{11, 14, 23, 24, 32, 33, 41, 42\}, & \{12, 13, 21, 22, 31, 34, 43, 44\}, \\
\{13, 14, 23, 24, 33, 34, 43, 44\}, & \{11, 12, 21, 22, 31, 32, 41, 42\}, \\
\{12, 13, 22, 23, 32, 33, 42, 43\}, & \{11, 14, 21, 24, 31, 34, 41, 44\}.
\end{array}$$

We notice that each of these is the set of vertices on a pair of adjacent parallel lines in G_M, a: these are all such pairs of lines.

Now we claim that the rank 2 poset P whose $X_2=X$, the vertex set of G_M, $X_1=\{s_1, s_2, \cdots, s_{1}$ and $X_0=\{0\}$, is the one attachable to G_M. Suppose Y is a 1-design in G_M of size 4, then λ_Y= from the way we construct P. For any design Y of size 6, we can easily check tl $|Y\cap C(s_j)|=3$ for each $j=1, 2, \cdots, 12$. We can also easily see that $\lambda_{\mathscr{F}_i}=i+2$ for $i=2, 3, 4$. F the converse, we need to show that any subset Y for which $\lambda_Y(s_j)$ is constant for all j forms 1-design in G_M. It is clear that there is no subset Y of X that satisfies $|Y\cap C(s_j)|=1$ for j $(j=1, 2, \cdots, 12)$. Since $|C(s_j)|=8$ for all j the non-existance of a subset Z with $\lambda_Z=7$ follo rom the nonexistance of Y with $\lambda_Y=1$. Suppose there is a subset Y with $\lambda_Y=8$, then Y must e X, which is the trivial design. Suppose Y is a subset with $\lambda_Y=2$, then Y must contain two lements from each $C(s_j)$, in other words, two vertices from each pair of adjacent parallel lines ı G_M. They are precisely those in $\mathcal{F}_0$. Similarly, we can see that any subset Y with $\lambda_Y=3$ elongs to $\mathcal{F}_1$. The remaining cases are automatic, and it completes our proof.

Before we close this section, we note the following:

1. From the proof of the theorem, the assumption $\lambda_{\mathcal{F}_0}=2$ implies $\gamma=8$ and $\lambda_{\mathcal{F}_i}=i+2$ for $i=1, 2, 3, 4$. In fact, assuming one of the constant $\lambda_{\mathcal{F}_1}=i+2$ we can get the same results.
2. This is the unique attachable poset with $\gamma=8$ and $\beta=6$, (β exists means $\lambda_{\mathcal{F}_i}$ exists for each i by Lemma 3.4.)
3. This poset has the following properties:
   a. $C(s_j)$ forms an anti 1-design, i.e., in the dual distribution of $C(s_j)$, $b_1=0$ and $b_2=0$, for all j.
   b. $|\{s\in X_1 : s\leq x,\ s\leq y\}|=4$ or 2 according as $(x,y)\in R_1$ or R_2.
4. There are another posets attachable to G_M with $\gamma=8$, for instance we can have one by deleting arbitrary one (or possibly a few) element from X_1 of P.
5. It is known that the Cartesian product of two Hamming graphs $H(n_1, q)$ and $H(n_2, q)$ is $H(n_1+n_2, q)$, and the product of r-copies of G_M, is also a distance regular graph having the parameters of $H(2r, 4)$, but non-isomorphic with $H(2r, 4)$ [8, 12]. It is easy to see that the product of two posets of $H(n_1, q)$ and $H(n_2, q)$ (with the usual product of posets) is the same as the poset of $H(n_1+n_2, q)$ with the ones we illustrated in Section 2. However, it is not known yet whether the r-th power of the poset for G_M is the right poset of the r-th power of G_M.

4. The Exceptional Johnson Graphs

: is known that there are three non-isomorphic exceptional graphs which have the same para- ers as $J(8,2)$ by Chang [5], and those are the only exceptional cases of Johnson graphs by williger [14]. We denote these three graphs by J_1, J_2, and J_3 (their adjacency matrices are ·n in Appendix). In this work we shall show that none of these has a regular attachable poset. say that a poset is ***regular*** if λ_Y is constant for all design Y with the same size. As we have : in Section 3, we start with classifying all 1-designs. Since the proof of the following lemma ıst analogue of that of Lemma 3.1, we shall omit the proof by simply giving the common ·section matrix $B(B_1)$ and the second eigenmatrix Q;

$$B=\begin{bmatrix}0 & 1 & 0\\ 12 & 6 & 4\\ 0 & 5 & 8\end{bmatrix}, \qquad Q=\begin{bmatrix}1 & 7 & 20\\ 1 & 7/3 & -10/3\\ 1 & -7/3 & 4/3\end{bmatrix}.$$

mma 4.1. *Let Y be a 1-design of the graph J, $J=J_1$, J_2, J_3, and let $\underline{a}=(a_0, a_1, a_2)$ be the · distribution of Y with respect to J, as usual. Then the type $(|Y|;\ a_1, a_2)$ of Y must be $m-2, m+1)$ for some m, $m=2, 3, \cdots, 14$.*

: note that each J does not necessarily have every such type of designs. However, all three graphs, as well as $J(8,2)$, have 1-designs of size 4.

Theorem 4.2. *None of J_1, J_2, and J_3 has a nontrivial attachable regular poset.*

Proof. Let $J=J_1$, and $X=\{1,2,\cdots,28\}$ be the vertex set of J. Let $\mathscr{F}$ be the family of 1-designs of size 4 (for the list see Appendix). Suppose we assume that there is a regular poset P. Then each element $s\in X_1$, $|C(s)\cap Y|$ must be constant for all $Y\in\mathscr{F}$. We denote the constant by λ, then λ would be either 1, or 2, or 3, or 4. Suppose $\lambda=4$, then the poset is the trivial one. Suppose $\lambda=1$ and $s\in X_1$, then $C(s)=\{x\in X: x\geq s\}$ contains exactly one element from each $Y\in\mathscr{F}$. Without loss of generality, let the vertex 1 belong to $C(s)$. Then $C(s)$ must be contained in the set $\{1,2,\cdots,13\}$ $(=\{x\in X: d(x,1)\leq 1\})$ because the complement of the set is contained in the union of all 1-designs consisting the vertex 1. Obviously, $C(s)$ can not be $\{1\}$ because there are designs which do not consist 1, so we assume $a\in C(s)$ for some $a\in\{2,3,\cdots,13\}$. Then $C(s)$ must be contained in $\{x\in X: d(x,1)\leq 1,\ d(x,a)\leq 1\}$ from the our assumption $\lambda=1$, and thus, $|C(s)|\leq 8$ $(p_1{}^1{}_1=6)$. Now it is easy to see that there is at least one design $Y\in\mathscr{F}$ which does not intersect with $C(s)$. We conclude that λ can not be 1, and it is also obvious that λ can not be 3. Suppose now $\lambda=2$, then, in particular, for the design $\{1,14,25,28\}$, $|C(s)\cap\{1,14,25,28\}|=2$. This is true for an $s\in X_1$. Suppose there is an $s\in X_1$ such that $C(s)\cap\{1,14,25,28\}=\{1,14\}$. (We show that X_1 has i fact no such s.) Then from the design $\{1,15,23,28\}$ and $\{1,16,21,25\}$, $C(s)$ should have one fro $\{15,23\}$ and one from $\{16,21\}$. Suppose we assume that 15 and 16 belong to $C(s)$, then $C(s)$ ca have none of 9, 10, 11, and 12, from the designs $\{9,12,15,16\}$ and $\{10,11,15,16\}$. This impli that 17 and 8 belong to $C(s)$ from the designs $\{10,11,14,17\}$ and $\{8,9,14,25\}$. From the desig $\{8,9,15,23\}$ and $\{8,11,14,19\}$, $C(s)$ can not have 19, nor 23. Moreover, from the design $\{5,1$(18,23\}$ $C(s)$ must have 5 and 18, and can not have 13 because of the design $\{5,13,16,20\}$. follows a contradiction because $|C(s)\cap Y|=1$ for the design $Y=\{2,13,19,23\}\in\mathscr{F}$. It forces us take a look at the other choices rather than $\{15,16\}$ for the intersection $C(s)$ with the designs { 15, 23, 28} and $\{1,16,21,25\}$. However, it is easy to see that we will have the same contradictic Similarly, we can check that there is no such $s\in X_1$ that satisfies $|C(s)\cap\{1,14,25,28\}|=2$. Tl tells us there is no attachable regular poset except the trivial ones. For the graph J_2 and J_3, it c be shown analogously.

We note that it is not known if there is any non-regular poset which characterizes the desi in any of the graphs J_1, J_2, or J_3.

Acknowledgments: I wish to thank my advisor Professor Eiich Bannai for suggesting this ɪ blem and for his valuable advice.

5. Appendix

1. The adjacency matrix of J_1 [5].

$$
\left(\begin{array}{cccccccccccccccccccccccccccc}
0&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&1&1&1&1&1&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
1&1&0&1&1&1&0&0&1&1&0&0&0&1&1&0&0&0&1&1&1&0&0&0&0&0&0&0\\
1&1&1&0&1&1&0&0&1&0&1&0&0&1&0&1&0&0&0&0&0&1&1&1&0&0&0&0\\
1&1&1&1&0&0&1&0&1&0&1&0&0&0&1&0&1&0&1&0&0&1&0&0&1&0&0&0\\
1&1&1&1&0&0&0&1&0&1&0&1&0&1&0&1&0&0&0&1&0&0&1&0&0&1&0&0\\
1&1&0&0&1&0&0&1&0&0&1&1&1&0&1&0&1&1&0&0&0&0&0&0&1&0&1&0\\
1&1&0&0&0&1&1&0&0&1&0&1&1&0&0&1&1&1&0&0&0&0&0&0&0&1&0&1\\
1&0&1&1&1&0&0&0&0&1&1&0&1&0&0&0&0&0&1&0&1&1&0&1&0&0&0&1\\
1&0&1&0&0&1&0&1&1&0&0&1&1&0&0&0&0&0&1&1&1&0&0&0&0&1&0&1\\
1&0&0&1&1&0&1&0&1&0&0&1&1&0&0&0&0&0&0&0&0&1&1&1&1&0&1&0\\
1&0&0&0&0&1&1&1&0&1&1&0&1&0&0&0&0&0&0&1&0&0&1&0&1&1&1&0\\
1&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&1&0&0&1&0&0&1&0&0&1&1\\
0&1&1&1&0&1&0&0&0&0&0&0&0&0&1&1&0&1&0&1&1&0&1&1&0&0&1&0\\
0&1&1&0&1&0&1&0&0&0&0&0&0&1&0&0&1&1&1&1&1&0&0&0&1&0&1&0\\
0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&0&1&1&0&0&0&1&1&1&0&1&0&1\\
0&1&0&0&1&0&1&1&0&0&0&0&0&0&1&1&0&1&1&0&0&1&0&0&1&1&0&1\\
0&1&0&0&0&0&1&1&0&0&0&0&1&1&1&1&1&0&0&0&1&0&0&1&0&0&1&1\\
0&0&1&0&1&0&0&0&1&1&0&0&0&0&1&0&1&0&0&1&1&1&0&0&1&1&0&1\\
0&0&1&0&0&1&0&0&0&1&0&1&0&1&1&0&0&0&1&0&1&0&1&0&1&1&1&0\\
0&0&1&0&0&0&0&0&1&1&0&0&1&1&1&0&0&1&1&1&0&0&0&1&0&0&1&1\\
0&0&0&1&1&0&0&0&1&0&1&0&0&0&0&1&1&0&1&0&0&0&1&1&1&1&0&1\\
0&0&0&1&0&1&0&0&0&0&1&1&0&1&0&1&0&0&0&1&0&1&0&1&1&1&1&0\\
0&0&0&1&0&0&0&0&1&0&1&0&1&1&0&1&0&1&0&0&1&1&1&0&0&0&1&1\\
0&0&0&0&1&0&1&0&0&0&1&1&0&0&1&0&1&0&1&1&0&1&1&0&0&1&1&0\\
0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&1&1&0&1&1&0&1&1&0&1&0&0&1\\
0&0&0&0&0&0&1&0&0&0&1&1&1&1&1&0&0&1&0&1&1&0&1&1&1&0&0&0\\
0&0&0&0&0&0&0&1&1&1&0&0&1&0&0&1&1&1&1&0&1&1&0&1&0&1&0&0
\end{array}\right)
$$

2. The adjacency matrix of J_2 [5].

$$
\begin{pmatrix}
0&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&1&1&1&1&1&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
1&1&0&1&1&0&0&0&1&1&1&0&0&1&1&1&0&0&1&1&0&0&0&0&0&0&0&0\\
1&1&1&0&1&1&0&0&1&0&0&1&0&1&1&0&0&0&0&0&1&1&1&0&0&0&0&0\\
1&1&1&1&0&0&1&0&0&1&0&0&1&1&1&0&0&0&0&0&1&0&0&1&1&0&0&0\\
1&1&0&1&0&0&1&1&1&0&0&1&0&0&0&0&1&1&0&0&0&1&1&0&0&1&0&0\\
1&1&0&0&1&1&0&1&0&1&0&0&1&0&0&0&1&1&0&0&0&0&0&1&1&1&0&0\\
1&1&0&0&0&1&1&0&0&0&1&1&1&0&0&1&1&1&0&0&0&0&0&0&0&0&1&1\\
1&0&1&1&0&1&0&0&0&1&1&1&0&0&0&0&0&0&1&1&0&1&1&0&0&1&0&0\\
1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&1&0&0&0&1&1&1&0&0\\
1&0&1&0&0&0&0&1&1&1&0&1&1&0&0&1&0&0&1&1&0&0&0&0&0&0&1&1\\
1&0&0&1&0&1&0&1&1&0&1&0&1&0&0&0&0&0&0&0&1&1&1&0&0&0&1&1\\
1&0&0&0&1&0&1&1&0&1&1&1&0&0&0&0&0&0&0&0&1&0&0&1&1&0&1&1\\
0&1&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&1&0&1&1&0&1&0&0&1&0\\
0&1&1&1&1&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&1&0&1&0&1&0&0&1\\
0&1&1&0&0&0&0&1&0&0&1&0&0&1&1&0&1&1&1&1&0&0&0&0&0&0&1&1\\
0&1&0&0&0&1&1&1&0&0&0&0&0&1&0&1&0&1&1&0&0&1&0&1&0&1&1&0\\
0&1&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&1&0&0&1&0&1&1&0&1\\
0&0&1&0&0&0&0&0&1&1&1&0&0&1&0&1&1&0&0&1&0&1&0&1&0&1&1&0\\
0&0&1&0&0&0&0&0&1&1&1&0&0&0&1&1&0&1&1&0&0&0&1&0&1&1&0&1\\
0&0&0&1&1&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&1&1&1&1&0&1&1\\
0&0&0&1&0&1&0&0&1&0&0&1&0&1&0&0&1&0&1&0&1&0&1&1&0&1&1&0\\
0&0&0&1&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&1&1&1&0&0&1&1&0&1\\
0&0&0&0&1&0&1&0&0&1&0&0&1&1&0&0&1&0&1&0&1&1&0&0&1&1&1&0\\
0&0&0&0&1&0&1&0&0&1&0&0&1&0&1&0&0&1&0&1&1&0&1&1&0&1&0&1\\
0&0&0&0&0&1&1&0&1&1&0&0&0&0&0&0&1&1&1&1&0&1&1&1&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0&1&1&1&1&0&1&1&0&1&0&1&1&0&1&0&0&0&1\\
0&0&0&0&0&0&0&1&0&0&1&1&1&0&1&1&0&1&0&1&1&0&1&0&1&0&1&0
\end{pmatrix}
$$

3. The adjacency matrix of J_3 [5].

$$
\left(\begin{array}{cccccccccccccccccccccccccccc}
0&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&1&1&1&1&1&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
1&1&0&1&1&0&0&0&1&1&1&0&0&1&1&1&0&0&1&1&0&0&0&0&0&0&0&0\\
1&1&1&0&0&1&0&0&1&0&0&1&1&1&1&1&0&0&0&0&1&1&0&0&0&0&0&0\\
1&1&1&0&0&0&1&1&0&1&1&0&0&1&0&0&1&0&1&0&0&0&1&1&0&0&0&0\\
1&1&0&1&0&0&1&1&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0&0&1&1&0&0\\
1&1&0&0&1&1&0&1&0&1&0&1&0&0&0&0&1&1&0&0&0&0&1&0&1&0&1&0\\
1&1&0&0&1&1&1&0&0&0&1&0&1&0&0&0&1&1&0&0&0&0&0&1&0&1&0&1\\
1&0&1&1&0&0&0&0&0&1&1&1&1&0&0&1&0&0&0&1&0&1&0&0&0&0&1&1\\
1&0&1&0&1&0&1&0&1&0&1&1&0&0&0&0&0&0&1&1&0&0&1&0&1&0&1&0\\
1&0&1&0&1&0&0&1&1&1&0&0&1&0&0&0&0&0&1&1&0&0&0&1&0&1&0&1\\
1&0&0&1&0&1&1&0&1&1&0&0&1&0&0&0&0&0&0&0&1&1&1&0&1&0&1&0\\
1&0&0&1&0&1&0&1&1&0&1&1&0&0&0&0&0&0&0&0&1&1&0&1&0&1&0&1\\
0&1&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&1&0&1&1&1&1&0&0&0&0\\
0&1&1&1&0&1&0&0&0&0&0&0&0&1&0&1&0&1&1&1&1&0&0&0&1&1&0&0\\
0&1&1&1&0&0&0&0&1&0&0&0&0&1&1&0&1&1&0&1&0&1&0&0&0&0&1&1\\
0&1&0&0&1&0&1&1&0&0&0&0&0&1&0&1&0&1&0&0&0&1&1&1&0&0&1&1\\
0&1&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&1&0&0&0&0&1&1&1&1\\
0&0&1&0&1&0&0&0&0&1&1&0&0&1&1&0&0&0&0&1&1&0&1&1&1&1&0&0\\
0&0&1&0&0&0&0&0&1&1&1&0&0&0&1&1&0&1&1&0&0&0&0&0&1&1&1&1\\
0&0&0&1&0&1&0&0&0&0&0&1&1&1&1&0&0&0&1&0&0&1&1&1&1&1&0&0\\
0&0&0&1&0&0&0&0&1&0&0&1&1&1&0&1&1&0&0&0&1&0&1&1&0&0&1&1\\
0&0&0&0&1&0&1&0&0&1&0&1&0&1&0&0&1&0&1&0&1&1&0&1&1&0&1&0\\
0&0&0&0&1&0&0&1&0&0&1&0&1&1&0&0&1&0&1&0&1&1&1&0&0&1&0&1\\
0&0&0&0&0&1&1&0&0&1&0&1&0&0&1&0&0&1&1&1&1&0&1&0&0&1&1&0\\
0&0&0&0&0&1&0&1&0&0&1&0&1&0&1&0&0&1&1&1&1&0&0&1&1&0&0&1\\
0&0&0&0&0&0&1&0&1&1&0&1&0&0&0&1&1&1&0&1&0&1&1&0&1&0&0&1\\
0&0&0&0&0&0&0&1&1&0&1&0&1&0&0&1&1&1&0&1&0&1&0&1&0&1&1&0
\end{array}\right)
$$

4. The list of 1-designs of size 4 in J_1

{1, 14, 25, 28}, {1, 15, 23, 28}, {1, 15, 24, 26}, {1, 16, 19, 27},
{1, 16, 21, 25}, {1, 17, 20, 24}, {1, 17, 21, 23}, {1, 18, 19, 23},
{1, 18, 20, 22}, {2, 9, 26, 27}, {2, 10, 22, 27}, {2, 10, 24, 25},
{2, 11, 20, 28}, {2, 11, 21, 26}, {2, 12, 19, 24}, {2, 12, 21, 22},
{2, 13, 19, 23}, {2, 13, 20, 22}, {3, 7, 23, 28}, {3, 7, 24, 26},
{3, 8, 22, 27}, {3, 8, 24, 25}, {3, 11, 18, 26}, {3, 12, 17, 24},
{3, 12, 18, 22}, {3, 13, 16, 25}, {3, 13, 17, 23}, {4, 7, 20, 27},
{4, 7, 21, 26}, {4, 8, 19, 27}, {4, 8, 21, 25}, {4, 10, 17, 27},
{4, 10, 18, 25}, {4, 12, 15, 28}, {4, 12, 17, 21}, {4, 12, 18, 19},
{4, 13, 15, 26}, {4, 13, 17, 20}, {5, 6, 27, 28}, {5, 8, 20, 24},
{5, 8, 21, 23}, {5, 10, 16, 27}, {5, 10, 18, 23}, {5, 12, 14, 28},
{5, 12, 16, 21}, {5, 13, 14, 26}, {5, 13, 16, 20}, {6, 7, 19, 24},
{6, 7, 21, 22}, {6, 9, 17, 27}, {6, 9, 18, 25}, {6, 11, 15, 28},
{6, 11, 17, 21}, {6, 11, 18, 19}, {6, 13, 15, 22}, {7, 9, 14, 26},
{7, 9, 16, 20}, {7, 10, 14, 22}, {8, 9, 14, 25}, {8, 9, 15, 23},
{8, 11, 14, 19}, {9, 12, 14, 17}, {9, 12, 15, 16}, {10, 11, 14, 17},
{10, 11, 15, 16}

References

1. E. Bannai and T. Ito, *Algebraic Combinatorics I: Association Schemes*, Benjamin Cumming Lecture Note Series, Menlo Park, CA, 1984.
2. E. Bannai and T. Ito, *Algebraic Combinatorics II*, (in preperation).
3. N.L. Biggs, *Algebraic Graph Theory*, Cambridge Univ. Press, Cambridge, 1974.
4. G. Birkhoff, *Lattice Theory*, American Mathematical Society Colloquium Publications, Vol.25 Amer. Math. Soc., Providencee, R.I., 1973.
5. L.C. Chang, *Association Schemes of Patially Balanced Designs with Parameters $v=28$ $n_1=12$, $n_2=15$, and $p_1{}^2{}_1=4$*, Sci. Record, Math. New Ser. Vol. 4, No. 1 (1960), 12~18.
6. P. Delsarte, *An Algebraic Approach to the Association Schemes of Coding Theory*, Philij Res. Repts. Suppl. 1973, No. 10.
7. P. Delsarte, *Association Schemes and t-Designs in Regular Semi-lattices*, J. Combin. Theo: Ser. A 20 (1976), 230~243.
8. M. Doob, *On Graph Products and Association Schemes*, *Utilitas Mathematica*, Vol. (1972), 291~302.
9. Y. Egawa, *Characterization of $H(n, q)$ by the Parameters*, J. Combin. Theory Ser. A (1981), 108~125.

10. A. Munemasa, ***An Analogue of t-Designs in the Association Schemes of Alternating Bilinear Forms***, Graphs and Combinatorics (to appear).

11. S.S. Shrikhande, ***The Uniqueness of the L_2 Association Scheme***, Ann. Math. Statist. 30 (1959), 781～798.

12. S.Y. Song, ***Products of Distance Regular Graphs***, Utilitas Mathematica (to appear).

13. D. Stanton, ***t-designs in Classical Association Schemes***, Graphs and Combinatorics (to appear).

14. P. Terwilliger, *The Johnson Graph $J(d,n)$ is Unique if $(d,n) \neq (2,8)$*, Discrete Mathematics 58 (1986), 175～189.