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On Universal Derivation Modules*

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——Dedicated to Professor Han Shick Park on his 60th birthday——

1. Introduction

There are two different notions of derivation modules of algebras. One is defined only for commutative algebras that are discussed in many papers [3], [4] and [8]. The orther is defined for algebras that are not necessarily commutative which appeared in [2]. These two notions are different in the sense that universal derivation module in [2] is not a generalization of the usual one defined for any commutative algebra.

Let R be a commutative ring with unity, A a unitary R-algebra and M an A-bimodule. A pair (M,d), where $d: A \longrightarrow M$ is an R-derivation, i.e., an R-linear map such that $d(ab)=ad(b)+d(a)b$ for $a, b \in A$, is called a derivation module of A. A derivation module homomorphism $f:(M,d) \longrightarrow (N,\delta)$, where (M,d), (N,δ) are derivation modules of A, is an A-bimodule homomorphism such that $f \circ d = \delta$. A derivation module (U,d) is called a universal derivation module of A if for any derivation module (M,δ), there exists a unique derivation module homomorphism $f:(U,d) \longrightarrow (M,\delta)$. A universal derivation module of A exists in the category of derivation modules of A, and is unique up to derivation module isomorphisms. In fact, there are two different constructions of them that are of interest. One is given in [2], the other by using a tensor product of three copies of A.

Let $U = A \otimes_R A \otimes_R A / J$, where J is the submodule of $A \otimes_R A \otimes_R A$ generated by $1 \otimes ab \otimes 1 - a \otimes b \otimes 1 - 1 \otimes a \otimes b$ for $a, b \in A$, and define $d: A \longrightarrow U$ by $d(a) = \Pi(1 \otimes a \otimes 1) + J$. Then it is easy to show that (U,d) is a universal derivation module of A. On the other hand, let $\pi: A \otimes_R A \longrightarrow A$ be the R-algebra homomorphism given by $\pi(\sum a_i \otimes b_i) = \sum a_i b_i$ for $a_i, b_i \in A$, and $U = ker\, \pi$. Then (U,d), where $d: A \longrightarrow U$ is an R-derivation given by $d(a) = 1 \otimes a - a \otimes 1$ for $a \in A$, is a universal derivation module of A. These two types have their uses according to the nature of problems.

2. Free Joins of Algebras

An R-algebra A is called a *free join* of a family $(A_\alpha)_{\alpha \in I}$ of its subalgebras if for any R-algebra

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C and a family $(f_\alpha)_{\alpha\in I}$ the $f_\alpha : A_\alpha \longrightarrow C$ of algebra homomorphisms, there exists a unique algebra homomorphism $f : A \longrightarrow C$ extending each $f_\alpha, \alpha\in I$. For example [3], let $(M_\alpha)_{\alpha\in I}$ be a family of submodules of an R-module M such that $M=\oplus_{\alpha\in I}M_\alpha$. Then the tensor algebra $T(M)$ of M is a free join of the family $(T(M_\alpha))_{\alpha\in I}$ of tensor algebras $T(M_\alpha)$, $\alpha\in I$.

Lemma 2. 1. *Let A be a free join of a family $(A_\alpha)_{\alpha\in I}$ of its subalgebras. If $A=\oplus_{\alpha\in I}A_\alpha$, $g_\alpha : A_\alpha \longrightarrow R$ is an R-algebra homomorphism for $\alpha\in I$, then for any finite sequence $\alpha_1, \cdots, \alpha_n$ of I such that $\alpha_i\neq\alpha_j$ for $i,j=1,\cdots,n$, $A_{\alpha_1}\cdots A_{\alpha_n}$ is isomorphic to $A_{\alpha_1}\otimes_R\cdots\otimes_R A_{\alpha_n}$.*

Proof. Let $(f_\alpha)_{\alpha\in I}$ be a family of algebra homomorphisms such that for α_i, $f_{\alpha_i} : A_{\alpha_i} \longrightarrow A_{\alpha_1}\otimes_R\cdots\otimes_R A_{\alpha_n}$ is given by $f_{\alpha_i}(a)=\otimes a_t$, where $a_i=a$, $a_j=1$ for $j\neq i$, and $f_\alpha=g_\alpha$ for $\alpha\neq\alpha_1, \cdots, \alpha_n$. Let $f' : A \longrightarrow A_\alpha\otimes_R\cdots\otimes_R A_{\alpha_n}$ be the extending homomorphism of f_α.

Then $f'|A_{\alpha_1}\cdots A_{\alpha_n}$ is an isomorphism.

Let A be a free join of a family $(A_\alpha)_{\alpha\in I}$ of i-ts subalgebra, and B is a direct sum of A_α. Let $T(A_\alpha)$ and $T(B)$ be tensor algebras of A_α and B respectively. If $h_\alpha : T(A_\alpha) \longrightarrow A_\alpha$ is an algebra homomorphism extending the identity map $i_\alpha : A_\alpha \longrightarrow A_\alpha$ and $k : T(B) \longrightarrow A$ is the algebra homomorphism extending $h_\alpha, \alpha\in I$ then it is easy to show [3] that k is onto and $ker\, h$ is an ideal of $T(B)$ generated by $\sum_{\alpha\in I} ker\, h_\alpha$. From this, we have the following fact.

Lemma 2. 2. *Let $\alpha_1, \ldots, \alpha_n$ be a finite sequence of I which are all different. Then the R-linear map $f : A_{\alpha_1}\otimes_R\otimes\cdots\otimes_R A_{\alpha_n} \longrightarrow A_{\alpha_1}\cdots A_{\alpha_n}$ given by $a_1\otimes\cdots\otimes a_n \longrightarrow a_1\cdots a_k$ for $a_i\in A_{\alpha_i}$ is an R-module isomorphism.*

Theorem 2. 3. *Let A be a free join of a family $(A_\alpha)_{\alpha\in I}$ of its subalgebras, and (U_α, d_α) a universal derivation module of A_α, $\alpha\in I$. Then (U,d) is a universal derivation module of A where $U=\otimes_{\alpha\in I}(A\otimes_{A\alpha}U_\alpha\otimes_{A\alpha}A)$ and $D : A \longrightarrow U$ is the R-derivation defined by*
$D(\sum a_1\cdots a_n)=\sum(\sum_{i=1} a_1\cdots a_{i-1})\otimes d_{\alpha_i}(a_i)\otimes(a_{i+1}\cdots a_n)$.

Proof. Let D_{α_i} be the R-linearization of the R-multilinear map $\Phi : A_{\alpha 1}\times\cdots\times A_{\alpha_i} \longrightarrow U$ given by $\Phi(a_1, \cdots, a_n)=a_1\cdots a_{i-1}\otimes d_{\alpha_i}(a_i)\otimes a_{i+1}\cdots a_n$. Then the map $D : A \longrightarrow U$ given by $D(\sum a_1\ldots a_n)=\sum(\sum_{i=1}^{n} D_{\alpha_i}(a_1\ldots a_n)$ is an R-derivation. It is not hard to show that this map D is a universal derivation of A.

3. Extensions of Algebras

Let E be a unitary extension algebra of an R-algebra A with unity. An ideal I of A is said to be an (*two-sided*) *E-dense ideal* if $EI=IE=E$. Every ideal of R containing an E-dense ideal i also E-dense. Furthermore, if I and J are E-dense ideals of R, then IJ, JI and $I\cap J$ are also E-dens ideals.

An extension algebra E of A is called a (*two-sided*) *fractional extension* of A if for $p, q\in E$ there exist E-dense ideals I, J such that $pI, Jq\in A$. It is follows simply that A itself is E-dense For example, the matrix ring M of the form

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix},\ a, b\in Z/(p),\ p : \text{a prime}$$

is a two-sided quotient ring which is neither left nor right artinian [6]. Hence M has a two-sid quotient Z-algebra. When Q is a two-sided quotient algebra of an R-algebra A relative to a multiplicative subset S of A without zero-divisor, then Q is a fractional extension of A. An A-bimodule M is said to be E-torsion free if for every E-dense ideal I and I and $x \in M$, $Ix=0$ and $xI=0$ imply $x=0$. For example, every E-bimodule is an E-torsion free.

Lemma 3.1. *For a fractional extension E of A, let M be a E-torsion free A-bimodule, and $f: M \longrightarrow E \otimes_A M \otimes_A E$ an A-bimodule homomorphism given by $f(x)=1 \otimes x \otimes 1$ is one-to-one. Also, every A-bimodule homomorphism $f: M \longrightarrow N$ is an E-bimodule homomorphism.*

Proof. See proposition 4 in [1]

Lemma 3.2. *Let E be a fractional extension of A, and M an E-bimodule. If $d, \delta : E \longrightarrow M$ are R-derivation such that $d=\delta$ on A, then d is equal δ on E.*

An A-bimodule H is called an *injective hull* of an A-bimodule M if H is a left(right) injective hull of the left $A \otimes_R A^{op}$-module(right $A^{op} \otimes_R A$-module) M.

Lemma 3.3. *Let E be a fractional extension of an R-algebra A, and M an E-torsion free A-bimodule. Then an injective hull H of M is E-torsion free.*

Proof. Let $T_l(H)=\{x \in H \mid Ix=0$ for some E-dense ideal I of $A\}$. Then $T_l(H)$ is an A-bisubmodule of H. H is a maximal essential extension of M and $T_l(H)M=0$, and hence $T_c(H)=0$. By the same way, we know that $T_l(H)=\{x \in H \mid xI=0$ for some E-dense ideal I of $A\}$ is zero. Consequently, H is E-torsion free.

Let M be an E-torsion free A-bimodule, and I an E-dense ideal of A. Consider I and A as a left $A \otimes_R A^{op}$-module. Then there exists a unique module homomorphism $f: A \longrightarrow H$ extending an A-bimodule homomorphism $\Phi: I \longrightarrow M$. Indeed, this f is a map A into M. From this fact, we have the following lemma.

Lemma 3.4. *Let E be a fractional extension of A, and M an A-bimodule. If $f: M \longrightarrow E \otimes_A M \otimes_A E$ is an A-bimodule homomorphism given by $f(x)=1 \otimes x \otimes 1$ for $x \in M$, then for any R-derivation $d: A \longrightarrow M$, there exists a unique R-derivation $\delta: E \longrightarrow E \otimes_A M \otimes_A E$ such that $\delta \mid A = f \circ d$.*

Theorem 3.5. *Let E be a fractional extension of an R-algebra, and (U, d) a universal derivation module of A. For any A-bimodule homomorphism $f: U \longrightarrow E \otimes_A U \otimes_A E$ given by $f(x)=1 \otimes x \otimes 1$ for $x \in U$, let $\delta: E \longrightarrow E \otimes_A U \otimes_A E$ be the R-derivation in above lemma. Then (U, δ) is a universal derivation module of E.*

Proof. Let (V, τ) be a universal derivation module of E, and $\tau' = \tau \mid A$. Let Φ be the A-linearization of the A-multilinear map $\phi: E \times U \times E \longrightarrow M$ given by $\phi(p, x, q) = pq(x)q$, where $g: U \longrightarrow V$ is the unique A-bimodule such that $g \circ d = \tau$. It is easy to prove that Φ is the unique derivation module homomorphism such that $\Phi \circ \delta = \tau$ on E. This means (U, δ) is a universal derivation module of E.

Corollary 3.6. *Let E be a fractional extension of A. A universal derivation module of A is E-torsion free if and only if a universal derivation module of E is trivial.*

4. Free Algebras

An R-algebra A containing a monoid M is called a monoid algebra of M over R if for any R-algebra C and monoid homomorphism $f: M \longrightarrow C$, there exists a unique R-algebra homomorphism $g: A \longrightarrow C$ extending f. For an A-bimodule V, there exists a unique R-derivation $d: A \longrightarrow V$ extending the map $d': M \longrightarrow V$ given by $d'(u,v)=d'(u)v+ud'(v)$ for $u, v \in M$. By the same way we know that for a free monoid M on a set X and A-bimodule V, there exists a unique map $d: M \longrightarrow V$ extending a map $\Phi: X \longrightarrow V$ and $d'(uv)=ud'(v)+d'(u)v$ for $u, v \in M$. In fact, d' is a map defined by

$$d'(x_1, \cdots, x_n)=\sum_{i=1}^{n} x_1 \cdots x_{i-1} \Phi(x_i) x_{i+1} \cdots x_n.$$

Lemma 4.1. *Let $R[X]$ be a free R-algebra generated by a set X, and V on $R[X]$-bimodule. Then there exists a unique R-derivation $d: R[X] \longrightarrow V$ extending a map $\Phi: X \longrightarrow V$. Furthermore, every universal derivation module of $R[X]$ is generated by $d(X)$.*

Lemma 4.2. *Let U be an $R[X]$-bimodule, and $d: R[X] \longrightarrow U$ an R-derivation extending a map $\Phi: X \longrightarrow U$ with $\Phi(X)$ as a basis U. Then (U, d) is a universal derivation module of $R[X]$.*

Proof. For a derivation (M, δ) of $R[X]$, the map $f: U \longrightarrow M$ given by $f(\sum_{x \in X} a_x \Phi(x))=\sum_{x \in X} a_x \delta(x)$ for $a_x \in R[X] \otimes_R R[X]^{op}$ is the unique derivation module homomorphism such that $f \circ d=\delta$.

Theorem 4.3. *Let $U=\bigoplus_{x \in X} U_x$, where $U_x \cong R[X] \otimes_R R[X]$, and $\Phi_x: X \longrightarrow U_x$ a map by $\Phi_x(y)=1_x$ if $y=x$ and 0 if $y \neq x$. Then (U, d) is a universal derivation module of $R[X]$, where $d: R[X] \longrightarrow U$ is an R-derivation extending $\Phi=\sum_{x \in X} \Phi_x$. In particular, $(R[x] \otimes_R R[x], d)$ is a universal derivation of $R[x]$, where $d: R[x] \longrightarrow R[x] \otimes_R R[x]$ is the R-derivation given by $d(x^n)=\sum_{i=1}^{n} x^{i-1} \otimes x^{n-i}$ for indeterminate x.*

Next, we will study of universal derivation modules of exact sequence of algebras.

Theorem 4.4. *Let $A \xrightarrow{f} B \longrightarrow D$ be a exact sequence of algebras over R, and (U, d) a universal derivation modules of A. If $J=U(\ker f)+(\ker f)U+Ad(\ker f)+d(\ker f)A$, and $\partial: B \longrightarrow U/J$ is an R-derivation defined by $\partial(b)=d(a)+J$ for $a \in f^{-1}(b)$, then $(U/J, \partial)$ is a universal derivation module of B.*

Proof. Suppose (U, δ) is a universal derivation module of B. A B-bimodule V can be conside as an A-bimodule with $av=f(a)v$, $va=vf(a)$ for $a \in A$, $v \in V$.

Since $\delta \circ f: A \longrightarrow V$ is an R-derivation of A, there exists a unique derivation module homomor phism $g: U \longrightarrow V$ such that $\delta \circ f=g \circ d$. Indeed, g is onto. Consider U/J as a B-bimodule b $b(u+J)=au+J$, $(u+J)b=ua+J$ for some $a \in f^{-1}(b)$. Since $g: U \longrightarrow V$ is onto and $J \subseteq \ker g$ there exists an onto derivation module homomorphism $k: U/J \longrightarrow V$. By definition of universa derivation module, there exists a unique derivation module homomorphism $\Phi: U \longrightarrow U/J$ such tha $\Phi \circ \delta=\partial$. Hence k is one-to-one. We proved that U/J is isomorphic to V. Hence $(U/J, \partial)$ is universal derivation module of A.

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