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A Generalization of the Kuratowski Intersection Theorem

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——Dedicated to Professor Han Shick Park on his 60th birthday——

In complete metric spaces, the Cantor intersection theorem is the well-known fact. Kuratowski 2] first noticed that the Cantor intersection theorem characterizes the metric completeness and roved a generalized form of it by using the concept of the measure of noncompactness. Recently, ark and Rhoades [4] demonstrate the two basic properties which characterize the metric complete-ess and there is no need for proving various theorems which characterize the metric completeness.

Now we state some definitions. Let (X, d) be a complete metric space and $A \subseteq X$ a subset of X. he *Kuratowski measure* of noncompactness of A is defined by

$\alpha(A) = inf\{\varepsilon > 0 \mid A$ can be covered with a finite number of sets having diameter smaller than $\varepsilon\}$.

om the definition, it is clear that if $A \subseteq B$ then $\alpha(A) \leq \alpha(B)$. If $\alpha(A)=0$, then A is precompact. subset $A \subseteq X$ is called a *compactly closed* subset of X if $A \cap K$ is closed for each compact subset of X. Every closed subset of X is clearly compactly closed, and compactly closed sets need not closed. For any $r>0$ and $x \in X$, we denote $B_r(x) = \{y \in X \mid d(x,y) \leq r\}$.

As we mentioned before, Kuratowski [2] proved the following result, which is a generalization the well-known Cantor intersection theorem:

Theorem. *Let (X,d) be a complete metric space. If (A_n) $(n=1,2,\ldots)$ is a decreasing sequence nonempty closed subsets of X such that $\lim_{n\to\infty} \alpha(A_n)=0$, then $\bigcap_{n=1}^{\infty} A_n$ is nonempty and compact.*

n recent paper [1], Horvath gave a generalization of the above Theorem, and also gave some plications to fixed points and nonempty intersetion theorem for multivalued mappings in complete tric topological vector spaces. In fact, Horvath proved his basic Theorem 1 using Kuratowski's eorem.

n this paper, we give a further generalization of Horvath's Theorem, and also obtain some ollaries.

Ve begin with the following:

emma. *Let (X,d) be a complete metric space and (A_n) $(n=1,2,\ldots)$ be a decreasing sequence nonempty compactly closed subsets of X. If $\lim_{n\to\infty} \alpha(A_n)=0$, then $\bigcap_{n=1}^{\infty} A_n$ is nonempty and compact.*

Proof. Since $\alpha(\bigcap_{n=1}^{\infty} A_n)=0$, $\bigcap_{n=1}^{\infty} A_n$ is precompact. Therefore $\overline{\bigcap_{n=1}^{\infty} A_n}$ is compact. Since $\bigcap_{n=1}^{\infty} A_n$ is com-ly closed, $\bigcap_{n=1}^{\infty} A_n = \overline{\bigcap_{n=1}^{\infty} A_n} \cap (\bigcap_{n=1}^{\infty} A_n)$ is also compact. It remains to show that $\bigcap_{n=1}^{\infty} A_n$ is nonempty.

We may assume that each A_n is infinite. From the Definition of the measure of noncompactness : follows that we can choose $x_1 \in A_1$ and $B_{\alpha(A_1)+\varepsilon/2}(x_1)$ which contains an infinite number c ments of A_1 and also has an infinite number of elements of $A_2, A_3, \cdots$. Next, we choo: $x_2 \in A_2$ele and $B_{\alpha(A_2)+\varepsilon/2^2}(x_2)$ which contains an infinite number of elements of $A_2, A_3, \ldots$, ar $B_{\alpha(A_2)+\varepsilon/2^2}(x_2) \subseteq B_{\alpha(A_1)+\varepsilon/2}(x_1)$. Continuing this process, we obtain sequences of points $\{x_1, x_2, \cdots\}$ a sets $\{B_{\alpha(A_i)+\varepsilon/2^i}(x_i) \mid i=1, 2, \cdots\}$. It is clear that $d(x_i, x_j) \leq \alpha(A_i) + \frac{\varepsilon}{2^i}$ for all $j \geq i$ and so $\{x$ is a Cauchy sequence. Let $x_0 = \lim x_n$, and let $K = \{x_1, x_2, \ldots, x_n, \ldots, x_0\}$. Then K is a compact sub: of X with $A_n \cap K \neq \phi$ for each n. Now let $B_1 = A_1 \cap K$ and $B_k = A_k \cap B_{k-1}$ $(k \geq 2)$. Then each B_k nonempty closed (actually compact) subset of X. Furthermore, $B_{k+1} \subseteq B_k$ for all $k = 1, 2, \ldots$. The fore, by Kuratowski's Theorem, $\bigcap_{k=1}^{\infty} B_k \neq \phi$. Furthermore,

$$\bigcap_{k=1}^{\infty} B_k = \bigcap_{k=1}^{\infty} A_k \cap K \neq \phi.$$

Hence we have $\bigcap_{k=1}^{\infty} A_k \neq \phi$. This completes the proof.

Now we prove the following generalization of Horvath's Theorem:

Theorem 1. *Let (X, d) be a complete metric space and $(A_i)_{i \in I}$ be a family of compactly clo subsets of X having the finite intersection property.*

If $\inf_{i \in I} \alpha(A_i) = 0$, then $\bigcap_{i \in I} A_i$ is nonempty and compact.

Proof. Since $\inf_{i \in I} \alpha(A_i) = 0$, for each $n \in N$ we can choose $A_{i(n)}$ with $\alpha(A_{i(n)}) < \frac{1}{n}$. Der $C_k = \bigcap_{n=1}^{k} A_{i(n)}$. Then each C_k is nonempty compactly closed and $\alpha(C_k) < \frac{1}{k}$. Moreover, $C_{k+1} \subseteq C_k$ all $k = 1, 2, \ldots$. Applying the previous Lemma to (C_k), we obtain $\bigcap_{n=1}^{\infty} A_{i(n)} = \bigcap_{n=1}^{\infty} C_k$ is nonempty precompact. Let J be any nonempty finite subset of I, and define

$$B_J^1 = \bigcap_{i \in J} A_j \text{ and } B_J^k = \bigcap_{i \in J} \Big(A_i \cap \big(\bigcap_{n=1}^{k-1} A_{i(n)}\big) \Big) \quad (k \geq 2).$$

Then B_J^k is a nonempty compactly closed subset of X and $B_J^{k+1} \subseteq B_J^k$ for each $k = 1, 2, \ldots$. Furt more, $\alpha\ (B_J^{k+1}) < \frac{1}{k}$ for each $k = 1, 2, \ldots$. Therefore, by the previous Lemma, we obtain that $\bigcap_{k=1}^{\infty} B$ nonempty and precompact. Hence,

$$\phi \neq \bigcap_{k=1}^{\infty} B_J^k = \bigcap_{j \in J} A_j \cap \big(\bigcap_{n=1}^{\infty} A_{i(n)}\big) \subseteq \bigcap_{j \in J} A_j \cap \overline{\big(\bigcap_{n=1}^{\infty} A_{i(n)}\big)}.$$

Denote B_j by $A_j \cap \overline{(\bigcap_{n=1}^{\infty} A_{i(n)})}$. Then $(B_j)_{j \in I}$ is a family of compact subsets of having the finite tersection property, so we have $\bigcap_{j \in I} B_j \neq \phi$. Therefore,

$$\phi \neq \bigcap_{j \in I} B_j = \bigcap_{j \in J} A_j \cap \overline{\big(\bigcap_{n=1}^{\infty} A_{i(n)}\big)} \subseteq \bigcap_{j \in I} A_j,$$

and the proof is completed.

Corollary. ([1]) *Let (X, d) be a complete metric space, and (A_i) be a family of none closed subsets of X having the finite intersection property. If $\inf_{i \in I} \alpha(A_i) = 0$, then $\bigcap_{i \in I} A_i$ is none and compact.*

Using Theorem 1, we can obtain several consequences. In this place, by following [1], we only prove the following:

Theorem 2. *Let (X,d) be a complete metric space, and $f: X \longrightarrow R$ be lower semicontinuous on each compact subset of X. If $\inf_{x\in X} \alpha(\{y\in X | f(y)\leq f(x)\})=0$, then f is bounded below and there exists $x_0\in X$ such that $f(x_0)=\inf_{x\in X} f(x)$.*

Proof. Let $A_x=\{y\in X | f(y)\leq f(x)\}$, then $(A_x)_{x\in X}$ is a family of nonempty compactly closed subsets of X having the finite intersection property. Since $\inf_{x\in X} (A_x)=0$, by Theorem 1, there exists $x_0\in \bigcap_{x\in X} A_x$, i.e., $f(x_0)\leq f(x)$ for all $x\in X$. This completes the proof.

References

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