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A Note on Dunford-Pettis Operators

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——Dedicated to Professor Han Shick Park on his 60th birthday——

Abstract: In this paper we will investigate the relations between Dunford-Pettis operators and weakly compact operators. And we get a characterization of a Banach space with the RNP.

Let (Ω, Σ, μ) be a finite measure space and $L_1(\mu)$ be the set of classes of real valued μ-measurable unctions on Ω. X, Y denote real Banach spaces. An operator $T: X \longrightarrow Y$ is called an *weakly ompact operator* if T maps bounded subset of X into a relatively weakly compact subset of Y. In operator $T: L_1(\mu) \longrightarrow X$ is said to be a *Dunford-Pettis operator* if T maps weakly compact ets into relatively norm compact subsets of X. An operator $T: L_1(\mu) \longrightarrow X$ is said to be *Bochner resp. Pettis) representable* if there exists a Bochner integrable essentially bounded (resp. Pettis ntegrable and scalary essentially bounded) function $h: \Omega \longrightarrow X$ such that for every $f \in L_1(\mu)$, $T(f)$ = Bochner-$\int_\Omega fhd\mu$ (resp. Pettis-$\int_\Omega fhd\mu$). A Banach space X is said to have the *Radon-Nikodym property* (abbreviated as RNP) if for every finite measure space (Ω, Σ, μ) and every perator $T: L_1(\mu) \dashrightarrow X$, T is Bochner representable. If the T is Pettis representable then the X said to have the *weak Radon-Nikodym property* (WRNP).

Throughout this paper we confine ourselves to bounded linear operators. The notions not defined ere can be seen in (3, 4).

Since relatively weakly compact set is not always relatively norm compact set, there may be a eakly compact operator that is not a Dunford-Pettis operator. A Banach space X is said to have e *Dunford-Pettis property* if for each Banach space Y, every weakly compact linear operator $: X \longrightarrow Y$ is a Dunford-Pettis operator.

Recently Diestel published his eminent paper on Dunford-Pettis property (2). One of frequently ed fact is that $L_1(\mu)$ has the Dunford-Pettis property. So the following theorem is obvious.

Theorem 1. *If $T: L_1(\mu) \longrightarrow X$ is an weakly compact operator, then T is a Dunford-Pettis erator for every Banach space X.*

Here we have a question. For what Banach space X does Dunford-Pettis operator $T: L_1(\mu) \longrightarrow X$ ply weakly compact operator?

First, let's investigate if there exist Dunford-Pettis operator which is not weakly compact.

Proposition 2. *If $T: L_1(\mu) \longrightarrow X$ is Pettis representable, then T is a Dunford-Pettis operator.*

Proof. Let $T: L_1(\mu) \longrightarrow X$ be Pettis representable with kernel h.

Suppose $T: L_\infty(\mu) \longrightarrow X$, then its adjoint $T^*: X^* \longrightarrow L_\infty^*(\mu)$ is defined by $T^*x^* = x^*h$. Thus $T^*(x^*) \in L_1(\mu)$ and $T: L_1(\mu) \longrightarrow X$ is Dunford-Pettis operator if and only if the L_∞-bounded set $\{x^*h : \|x^*\| \leqq 1\}$ is relatively norm compact in $L_1(\mu)$. By weak*-compactness of the unit ball of X^*, every pointwise cluster point of $\{x^*h : \|x^*\| \leqq 1\}$ is contained in this set, and is therefore measurable. By fundamental theorem of Fremlin's, every sequence in the L_∞-bounded set $\{x^*h : \|x^*\| \leqq 1\}$ has an a. e. convergent subsequence. This means $\{x^*h : \|x^*\| \leqq 1\}$ is L_1-compact and T is a Dunford-Pettis operator.

The above proposition says that if X has the weak Radon-Nikodym property, then every bounded linear operator $T: L_1(\mu) \longrightarrow X$ is Pettis representable and hence is a Dunford-Pettis operator (5). But not all representable operators on $L_1(\mu)$ are weakly compact operators.

Example. Let $T: L_1(\mu) \longrightarrow l_1$ be a quotient map. i.e., T is linear, bounded and onto. Since $L_1(\mu)$ is seperable, it is topologically isomorphic to a suitable quotient space of l_1. This guarantees the existence of such a mapping. Since l_1 has the Radon-Nikodym property T is representable. So by the Proposition 2, T is a Dunford-Pettis operator. Let K be the unit ball of $L_1(\mu)$, then $T(K)$ is a bounded subset of l_1. Since l_1 is not reflexive, $T(K)$ is not a relatively weakly compact subset of l_1. Thus T is not an weakly compact operator.

The following is a well known lemma.

Lemma 3. *Every weakly compact operator on $L_1(\mu)$ is representable.*

In the following theorem we will show that if the range space is reflexive, then every Dunford-Pettis operator on $L_1(\mu)$ implies weakly compact operator.

Theorem 4. *Let X be a reflexive Banach space. Then every operator $T: L_1(\mu) \longrightarrow X$ is a Dunford-Pettis operator. Moreover the operator T is an weakly compact operator.*

Proof. If X is a reflexive Banach space, then X has the Radon-Nikodym property. So every $T: L_1(\mu) \longrightarrow X$ is representable, which implies that T is a Dunford-Pettis operator. If K is the unit ball of $L_1(\mu)$, then $T(K)$ is a bounded subset of X. Since bounded subset of a reflexive Banach space is relatively weakly compact, T is an weakly compact operator.

Since $L_1(\mu)$ has the Dunford-Pettis property, Theorem 4 states that if X is reflexive then every operator $T: L_1(\mu) \longrightarrow X$ is an weakly compact as well as a Dunford-Pettis operator. So from Bourgain's results (1), the following corollary is rather evident.

Corollary 5. *When X is reflexive, for an operator $T: L_1(\mu) \longrightarrow X$ the following assertion are equivalent.*

a) T is an weakly compact operator.

b) The restriction of T to $L_p(\mu)$ for some $p(1 \leqq p < \infty)$ is a compact operator.

c) The restriction of T to L_∞ is a compact operator.

It was suspected for a while, if every Dunford-Pettis operator from $L_1(\mu)$ into a Banach spac X is representable, is every bounded linear operator $T: L_1(\mu) \longrightarrow X$ Bochner representable? Bou gain (1) solved this problem affirmatively.

Proposition 6. *If a Banach space X fails the Radon-Nikodym property, then there is a Dunford-Pettis operator $T: L_1(\mu) \longrightarrow X$ which is not representable* (1).

Now we will show that the equivalence of Dunford-Pettis operator and weakly compact operator characterizes a Banach space with the Radon-Nikodym property.

Theorem 7. *If every Dunford-Pettis operator $T: L_1(\mu) \longrightarrow X$ implies weakly compact operator, then X has the Radon-Nikodym property.*

Proof. Let $T: L_1(\mu) \longrightarrow X$ be a Dunford-Pettis operator. Then the T is weakly compact and hence is representable by Lemma 3. So X has the Radon-Nikodym property by Proposition 6.

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