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A Note on RC-Convergence

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1. Introduction

The properties of S-closed space was defined from the characteristics of Semi-open set. The properties of S-closed spaces can be characterized with regular closed(rc) or regular open (ro) subsets of a space $(X, \mathcal{J})$. And rc-convergence structure used to charateristics of S-closed space. In this paper investigating the several properties of S-closed and rc-convergence.

2. Preliminaries

A Hausdorff space X is H-closed if and only if for every open cover $\{U_a \mid a \in A\}$, there exists a finite subfamily $\{U_{a_i} \mid i=1, 2, \cdots\cdots, n\}$ such that the union of there closures cover X. A subset V of topological space is semi-open if and only if $int(V) \subset V \subset cl(V)$. A topological space X is S-closed if and only if every semi-open cover of X has a finite uubcollection whose closure cover X. A function $f: X \rightarrow Y$ is said to be semi-continuous if and only if the inverse image of semi-open is semi-open. A topological space is extremally disconnected if the cloure of every open set is open.

The following are major properties obtained from the characteristics of S-closed spac For $(X, \mathcal{J})$ the following are equivalent.

i) X is S-closed

ii) any cover A of X by regular-closed sets has a finite subcover

iii) any family A of regular-open sets such that $\cap A = \phi$ contains a finite $B \subset A$ such that $\cap B - \phi$

iv) every filter base on X has an rc-accumulation point in X

v) every maximal filter base on X rc-converges

Defnition 1. rc-convergence structure for X is $A(x) = \{cl_x C \mid x \in cl_x C \text{ and } C \in \mathcal{J}\}$.

Definition 2. A filter base F on X rc-converges to $x \in X$ if $< A(x) > \subset < \mathcal{J} >$.

Throughout this paper $(X, \mathcal{J})$ and (Y, τ) denote simply X and Y respectively. For $(X, \mathcal{J})$, $(X, \mathcal{J}_s)$ denotes simply X generated by the regular-open subsets of X. Let $rc(X) - \{x \mid x \in X \text{ and } x \text{ is regular closed}\}$ and $ro(X) = \{x \mid x \in X \text{ and } x \text{ is regular open}\}$.

Seperation axioms are not used unless otherwise specified. Moreover, unless otherwise indicated, no extremally disconnected space is Hausdorff.

3. Theorems

Theorem 1. *For $(X, \mathcal{J})$, the following are equivalent:*

i) X is S-closed

ii) for every cover $A \subset rc(X)$ there exists finite subcover

iii) for every $A \subset ro(X)$ such that $\cap A = \phi$, there exists a finite $B \subset A$ such that $\cap A = \phi$

iv) every maximal filter base on X rc-converges

v) every filter base on X rc-accumulates

Proof. i)$\Leftrightarrow$iv) We have claim that S-convergence is equivalent to rc-convergence. Let a filterbase $\mathcal{J}$ s-converge to $x \in X$ and $R \in A(x)$. Then $R = cl_x C_1 \cap \cdots\cdots \cap cl_x C_n$. Now for each $cl_x C_i$, $i = 1, 2, \cdots\cdots, n$, there exists $F_i \in \mathcal{J}$ such that $F_i \subset cl_x C_i$. Hence there exists $F \in \mathcal{J}$ such that $F \subset \cap \{F_i\} \subset R$. On the other hand. assume that $\mathcal{J}$ rc-converges to $x \in X$ and $V \in so(X)$, $x \in V$. Then $cl_x V \in rc(X)$ implies that for each $V \in so(X)$ such that $x \in V$ there exists an $F \in \mathcal{J}$ such that $F \subset cl_x V$.

ii)$\Rightarrow$i) Let X is not S-closed. Then there exists a cover $A \subset so(X)$ such that has no finite proximate subcover. Thus $\{cl_x V \mid V \in A\} \subset rc(X)$ has no finite subcover. This contradicts to ii).

iv)$\Leftrightarrow$v) A maximal filter base rc-converges if and only if it rc-accumulates.

i)$\Rightarrow$ii), ii)$\Leftrightarrow$iii) These follow immediately from $rc(X) \subset so(X)$ and $ro(X) = \{X - x \mid x \in rc(X)\}$.

Theorem 2. *The semi-continuous surjection of a S-closed space onto any Hausdorff space is H-closed.*

Proof. Let $f: X \to Y$ be a semi continuous surjection and V an arbitrary open cover of Y. Then $f^{-1}(V_a)$ is a semi-open cover of X. By hypothesis, there exists a finite sub-family such that $\cap_{i=1}^{n} cl(f^{-1}(V_{a_i}) = X$. Notice that $\cup_{i=1}^{n} f^{-1}(V_{a_i})$ being dense in X implies $\cup_{i=1}^{n} f(V_{a_i}) = X$. Then

$Y = f(X) = f(\cup_{i=1}^{n} f^{-1}\ (V_{a_i})) \subset f(cl(\cup_{i=1}^{n} f^{-1}(V_{a_i}))) = cl(\cup_{i=1}^{n} V_{a_i}) = \cup_{i=1}^{n} cl(V_{a_i})$. Therefore, *$Y$ is H-closed.*

Corollary. *The semi-continuous surjection of an S-closed space onto any regular space is compact.*

Theorem 3. *If X is a S-closed regular space, then X is extremally disconneted.*

Proof. Suppose that X is not extremally disconnected. Then there exists a regular open set $O \subset X$ such that $cl(O) - O$ and X-$cl(O)$ are nonempty. Let $x \in cl(O) - O$. Then for every neighborhood V *of* x, $V \cap O \neq \phi$. Therefore, $F = \{(V \cap O)\}$ forms a filterbase in $cl(O)$. Since $cl(O)$ is *S-closed, F s-accumulates* to some point x_o *in* $cl(O)$. The filterbase also converges to x_o. We claim that $x \notin cl(O) - O$; for if it were, then $x_o \in X - O$ and every member of F would have to intersect $X - O$, an impossibility. Thus, $x_o \in O$. There exists an open set U such that $x_o \in U \subset cl(U) \subset O$ and $x \in X - cl(U)$. But since F converges to x_o there must exist a neighborhood V fo x, such that $(V \cap O) \subset X - cl(U)$. This then would imply that $(V\cap O)\cap cl(U)=\phi$, contradicting the fact that F *s-accumulate to* x_o. Therefore X is not extremally disconnected is false.

Corollary. *If X is a Hausdorff space, then X is extremally disconnected.*

Corollary. *Let X be a regular compact space, Then X is S-closed if and only if X is extremally disconnected.*

Theorem 4. *If $f: X\rightarrow Y$ is an irresolute surjection from an S-closed space X, then Y is S-closed.*

Proof. Let $\{V_a\}$ be a semi-open cover of Y. Then $\{f^{-1}(V_a)\}$ is a semi-open cover of X and has a finite subfamily such that $\cup_{i=1}^{n} cl(f^{-1}(V_{a_i}))=X$. Since $\cup_{i=1}^{n} f^{-1}(V_{a_i})$ is dense in X, $\cup_{i=1}^{n} f(V_{a_i})=X$. Therefore

$$Y=f(X)=f(\cup_{i=1}^{n} f(Va_i))=\cup_{i=1}^{n} Va_i\subset \cup_{i=1}^{n} Va_i.$$

Hence Y is *H-closed.*

Theorem 5. *The irresolute function from an S-closed Hausdorff space is closed.*

Proof. Let $f: X\rightarrow Y$ be an irresolute function from an *S-closed space Y. Let* $y\in cl(f(x))$ and $N(y)$ be the open neighborhood filterbase about Y. By filterbase $F=f^{-1}(N(y))$ has an *s-accumulation* point x. We claim that the filterbase $f(F)$ accumulates to $f(x)$ in the usual sense. Indeed, let V be any open set containing $f(x)$. Then $f^{-1}(V)$ is a semi-open set containing x and therefore every $W\in N(y)$, $f^{-1}(W)\in F$, *and* $f^{-1}(W)\cap cl(f^{-1}(V))\neq\phi$. But $int(f^{-1}(w))\cap int(f^{-1}(V))\neq\phi$. Therefore

$$\phi\neq f(int(f^{-1}(W))\cap int(f^{-1}(V)))\subset f(f^{-1}(W)\cap f^{-1}(V))\subset W\cap V.$$

Since W and V were arbitrarily chosen, we have that $f(F)$ accumulates to $f(x)$. But $f(F)$ is a finer filterbase than $N(y)$, hence $N(y)$ accumulates to $f(x)$. Since $N(y)$ obviously converges to y, by the property of Hausdorff space, $f(x)=y$. Hence $y\in f(X)$ and $f(X)$ is closed in Y.

Theorem 6. *An almost-open almost-continuous map $f: (X,\mathcal{J})\rightarrow(Y,\tau)$ preserves rc-convergence.*

Proof. Let $F\in rc(y)$. Then $f^{-1}(F)\in rc(X)$. Thus if $f(x)\in cl_Y V$ for arbitrary $V\in\tau$ then $f^{-1}(cl_Y V)=cl_X H$, $H\in\mathcal{J}$. Consequently, $f^{-1}(cl_Y V)\in A(x)$. Hence the result follows.

References

1. Travis Thompson, S-closed space, *Proc. Amer. Math. Soc.*, Vol. 60, 1976.
2. James E Joseph and Myung H. Kwack, On S-closed space, *Proc. Amer. Math. Soc.*, Vol. 80, No. 2, 1980.
3. W. F. Pfeffer, The spaces which contain an S-space, *Proc. Amer. Math. Soc.*, Vol. 85, No. 4, 1982.
4. Tsugunori Nogura, Countably Compact Extensions of Topological space, *Top. and its Application,* 15, 1985.
5. James Dugundji, *Topology,* Allyn and Bacon, Inc., Boston, 1970.
6. Stephen Willard, *General Topology,* Addison-Wesley Publishing Company, 1970.
7. John L. Kelley, *General Topolory,* D. Van Nostrand Company, Inc., 1955.