The existence of a lower solution of \( \frac{4(n-1)}{n-2} \Delta u + Ku \frac{n+2}{n-2} = 0 \)

on compact manifolds

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1. Introduction

A basic problem in Riemannian geometry is that of studying the set of curvature functions that a manifold possesses. In this generality there has been such a great deal of work [2,3,4,5]. However, in this paper we shall be concerned with the existence of a solution of

\[
(1.1) \quad \frac{4(n-1)}{n-2} \Delta u + Ku \frac{n+2}{n-2} = 0, \quad u > 0.
\]

Let \((M, g)\) be a Riemannian manifold of dimension \(n\) and \(K\) be a given function on \(M\). One may ask the following question: can we find a new metric \(g_1\) on \(M\) such that \(K\) is the scalar curvature of \(g_1\) and \(g_1\) is conformal to \(g\) (i.e., there exists \(u > 0\) on \(M\) such that \(g_1 = u^{n-2} g\))?

If \(M\) admits \(k \equiv 0\) as the scalar curvature of \(g\), then this is equivalent to the problem of solving the elliptic equation

\[
(1.1) \quad \frac{4(n-1)}{n-2} \Delta u + Ku \frac{n+2}{n-2} = 0, \quad u > 0,
\]

where \(\Delta\) is the Laplacian in the \(g\) metric.

In [4], J.L. Kazdan and F.W. Warner have studied the necessary conditions of the solvability of \((1.1)\), i.e., \(K\) changes sign and \(\bar{K} < 0\). In this paper we shall prove the existence of a lower solution of \((1.1)\).

2. Main result

Let \((M, g)\) be a compact connected manifold of dimension \(n\), which is not necessarily orientable. We denote the volume element of this metric by \(dV\), the gradient by \(\nabla\), and the mean value of a function \(f\) on \(M\) is written \(\bar{f}\), that is,

\[
\bar{f} = \frac{1}{\text{vol}(M)} \int_M f \, dV.
\]
We let $H_{s, p}(M)$ denote the Sobolev space of functions on $M$ whose derivatives through order $s$ are in $L_p$. The norm on $H_{s, p}(M)$ will be denoted by $\| \cdot \|_{s, p}$. The usual $L_2(M)$ inner product will be written $\langle , \rangle$.

It turns out that (1.1) is easier to analyze if we free it from geometry and consider instead

\[(2.1) \quad \Delta u + Hu^{a} = 0, \quad u > 0,\]

where $H$ is an arbitrary function and $a > 1$ is a constant.

**Lemma 1.** Let $\dim M \geq 3$ and $p > \dim M$. Then there exists a constant $C > 0$ such that for any $u \in H_{1, p}(M)$, $\| u \|_{2} \leq C \| u \|_{1, p}$.

**Proof.** See 2.22 in [5] or equation (3.8) in [3].

**Lemma 2.** If a positive solution $u$ of (2.1) exists and $H \equiv 0$, then $H$ must change sign and $\bar{H} < 0$.

**Proof.** See Lemma 2.5 and Prop. 5.3 in [4].

**Theorem.** (*Existence of a lower solution*)

Let $H(\not\equiv 0)$ belong to $C^\alpha(M)$ such that $H$ changes sign and $\bar{H} < 0$. Then there exists a lower solution $u > 0$ of (2.1).

**Proof.** Taking the change of variable $u = e^v$, $v$ satisfies

\[\Delta v + |\nabla v|^2 + He^{cv} = 0,\]

where $c = a - 1 > 0$ is a constant. We claim that there exists $v \in H_{1, p}$ such that $\Delta v + He^{cv} = 0$. For this claim, define a set of functions $B$ by $B = \{ v \in H_{1, p}(M) : \int_M He^{cv} \, dV = 0, \quad \bar{v} = 0 \}$.

Since $H$ changes sign, it is easy to see that $B$ is not empty. We shall minimize the functional

\[J(v) = \int_M |\nabla v|^2 \, dV = \| \nabla v \|_2^2 \quad \text{for} \quad v \in B.\]

Clearly $J \geq 0$. Let $b = \inf_{v \in B} J(v)$. Say $\{ v_n \} \subset B$ is a minimizing sequence, so $J(v_n) \downarrow b$. Because $B$ is not empty, there is some $v_0 \in B$. Let $b_1 = J(v_0)$. Then we can assume $J(v_n) \leq b_1$ for all $n$. Since $M$ is compact, the Hölder inequality implies that $v_n \in H_{1, 1}(M)$ for each $n$. But $\bar{v}_n = 0$, so the Poincaré inequality also implies that $\| v_n \|_1 \leq \text{constant} \times J(v_n) \leq \text{constant}$ for all $n$. Because the unit ball in any Hilbert space is weakly compact, we conclude that there is some $v \in H_{1, 1}(M)$ such that a subsequence of the $v_n$'s, which we relabel $v_n$, converges weakly to $v$. This implies that $\bar{v} = 0$.

Since $H_{1, 1}(M) \subset L_2(M)$ is compact (the Kondrakov's imbedding theorem there is some $v_0 \in L_2(M)$ such that $v_n \rightharpoonup v_0$ strongly to $v_0$. So $v_n \rightharpoonup v_0$ weakly in $L_2(M)$, i.e., $v_n = v_0$. For each $n$, $v_n \in H_{1, p}$. Let $\bar{a} = \inf_{v_n} \| v_n \|_{1, p}$. There exists a subsequence $\{ v_{n_k} \}$ of $v_n$'s such that $\| v_{n_k} \|_{1, p} \rightharpoonup \bar{a}$. Let $b_2 = \| v_{n_2} \|_{1, p}$. Then we may assume that $\| v_{n_k} \|_{1, p} \leq b_2$ for all $n_k$. Because $H_{1, p}$ is reflexive, the unit ball in $H_{1, p}$ is weakly sequentially compact, so we conclude that there is some $\bar{v} \in H_{1, p}$ such that $v_{n_k}$ converges $\bar{v}$ weakly.

Since $H_{1, p} \subset C^\alpha$ is compact for some small $\alpha > 0$, there is some $\bar{v}_0 \in C^\alpha(M)$ such that
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a subsequence of the \( v_{n_k} \)'s, which we relabel \( v_{n_k} \), converges strongly to \( \tilde{v}_o \). But \( \|v_{n_k} - \tilde{v}_o\|_p \leq \|v_{n_k} - v_o\|_p \leq \|v_{n_k} - v_o\|_{C^0} \) so \( v_{n_k} \rightarrow \tilde{v}_o \) strongly in \( L_p \). There \( \tilde{v}_o = \tilde{v} \). Since \( \|v_{n_k}\| \) is a subsequence of \( v_n 's \) and \( \|v_{n_k} - \tilde{v}_o\|_1 \leq \text{constant} \times \|v_{n_k} - \tilde{v}_o\|_p \), \( v_{n_k} \rightarrow \tilde{v}_o \) strongly in \( L_1(M) \).

Hence \( v = v_o = \tilde{v} = \tilde{v}_o \), i.e., \( v \in H_{1, \rho} \cap H_{1,1} \).

Combining these facts and using the inequality \( |e^t - 1| \leq |t| e^{|t|} \), we find that

\[
\|f\|H \|e^{cv} - e^{cv_n}\| \leq \int f |e^{cv} - e^{cv_n}| \leq \int f |e^{cv} - e^{cv_n}| \leq \|H\|_{\infty} \int |1 - e^{cv_n - cv}| \leq \|H\|_{\infty} \sup_{v \in B} |1 - e^{cv_n - cv}| \leq \|e^{cv} - e^{cv_n}\|_{C^0} \to 0
\]

Since \( \|v_n - v\|_{C^0} \to 0 \),

Hence \( \int f e^{cv} \, dV = 0 \) and \( v = 0 \), i.e., \( v \in B \).

To conclude that \( v \) minimizes \( J \) for all \( v \in B \), we use the general result that whenever \( v_n \) converge to \( v \) weakly in a normed space, \( \|v\| \leq \liminf \|v_n\| \). (See theorem 3.17 in 5)

Hence \( \|v\|_{1,1} \leq \lim inf \|v_n\|_{1,1} \). Since \( \tilde{H}_{1,1}(M) = \{ v \in H_{1,1}(M) : \tilde{v} = 0 \} \) is a Hilbert subspace, \( \mathcal{J}(v) \) is a norm equivalent to the norm \( \| \cdot \|_{1,1} \) on \( \tilde{H}_{1,1}(M) \). Therefore \( J(v) \leq \mathcal{J}(v_n) \) for all \( n \). Thus \( v \) minimizes \( J \) in \( B \).

Since \( v \) minimizes \( J \) in \( B \), by the standard Lagrange multiplier theory, we find that there are constants and such that for any \( \varphi \in H_{1, \rho}(M) \),

\[
\int [2\nabla v \varphi + \lambda e^{cv} \varphi + \mu \varphi] \, dV = 0.
\]

This is the Euler-Lagrange equation. Since \( H \subseteq C^\infty(M) \), by \( L^p \) regularity theory, \( \varphi \subseteq C^\infty(M) \). \( \varphi \equiv 1 \) gives \( \mu = 0 \). And \( \varphi = e^{-cv} \) and \( \tilde{H} < 0 \) show that \( \lambda < 0 \). So we can write \( \lambda = 2e^r \) for some constant \( r \). Then \( u = v + r \) is the desired solution \( u \in C^\infty(M) \) of \( \Delta u + He^{cu} = 0 \).

Thus \( u \) is the lower solution of \( \Delta u + Hu^a = 0 \), \( u > 0 \).

References