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A Simple Matrix Factorization Approach to Fast Hadamard Transform

(單純한 메트릭스 階乘接近에 의한 高速 아다마르變換)

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要 約

本 論文은 Hadamard 變換의 高速 알고리즘을 開發하기 위해 單純한 階乘 Hadamard 메트릭스를 제시했다. 이 分解된 메트릭스는 아이덴티디 메트릭스와 바로 앞단의 階乘된 Hadamard 메트릭스의 크로넥커積에 의해 表示된다. 여기서 Hadamard 메트릭스가 크로넥커積 및 階乘 메트릭스 方法에 의해 效率的으로 나타낼 수 있음을 數學的으로 證明했다.

Abstract

This paper presents a simple factorization of the Hadamard matrix which is used to develop a fast algorithm for the Hadamard transform. This matrix decomposition is of the kronecker products of identity matrices and successively lower order Hadamard matrices.

This following shows how the Kronecker product can be mathematically defined and efficiently implemented using a factorization matrix methods.

I. Introduction

The Hadamard transform has recently been applied in digital communication, the transmission of digital images, and also in pattern recognition for image processing and feature extraction.[1] [5] [7]

The elements of a Hadamard matrix take on values of plus and minus 1 only. This leads to simple implementation with electronic technology and simplifies the analysis by digital computer.

Furthermore, FHT can be used by factoring the Hadamard matrix.[1][2] This in turn reduces the number of required operation and provides a faster computer implementation.

This paper introduces a basic Hadamard matrix partition and then successive Kronecker product.[3] [4] The result of this method was easily shown to be the sparse matrix.[4] [6]

II. FHT Development

It is well known that the matrix factorization method has long been established.[3] Generally, to achieve (In place) computation, the existing methods require a shuffle right after each operation has been performed.

In order to apply the FHT directly on the lower Hamadard matrix ($[H]_2$) decomposition and still retain the property by Kronecker product use the following definition.

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def 1. $[H]_1=[1],\quad [H]_2=\begin{vmatrix} + & + \\ + & - \end{vmatrix}$ (1)

def 2. $[I]_1=[1],\quad [I]_2=\begin{vmatrix} + & 0 \\ 0 & + \end{vmatrix}$ (2)

The Hadamard matrix is represented by the sign for the matrix decomposition.

$$[H]_2=\begin{vmatrix} + & + \\ + & - \end{vmatrix}=\begin{vmatrix} + & 0 \\ 0 & + \end{vmatrix}\begin{vmatrix} + & + \\ + & - \end{vmatrix}=[I]_2[H]_2=[H]_2[I]_2 \quad (3)$$

where + and − indicate +1 and −1, respectively.

$$[H]_4=\begin{vmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{vmatrix}=\begin{bmatrix} + & + \\ + & - \end{bmatrix}\otimes\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}\times\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}\otimes\begin{bmatrix} + & + \\ + & - \end{bmatrix}$$

$$=[H]_2\otimes[I]_2\times[I]_2\otimes[H]_2$$

$$=([H]_2[I]_2)\otimes([I]_2[H]_2) \quad (4)$$

or

$$\begin{vmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{vmatrix}=\begin{bmatrix} + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & - & 0 \\ 0 & + & 0 & - \end{bmatrix}\begin{bmatrix} + & + & 0 & 0 \\ + & - & 0 & 0 \\ 0 & 0 & + & + \\ 0 & 0 & + & - \end{bmatrix} \quad (5)$$

$$[H]_8=[H]_2\otimes[I]_2\otimes[I]_2\times[I]_2\otimes[H]_4$$

$$=[H]_2\otimes[I]_2^{[2]}\times[I]_2\otimes[H]_2\otimes[I]_2\times[I]_2^{[2]}\otimes[H]_2$$

$$=\begin{bmatrix} + & + \\ + & - \end{bmatrix}\otimes\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}\otimes\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}\times\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}\otimes\begin{bmatrix} + & + \\ + & - \end{bmatrix}\otimes\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}\times\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}\otimes\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}\otimes\begin{bmatrix} + & + \\ + & - \end{bmatrix}$$

$$=\begin{bmatrix} + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & - & 0 \\ 0 & + & 0 & - \end{bmatrix}\otimes\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}\times\begin{bmatrix} + & + & 0 & 0 \\ + & - & 0 & 0 \\ 0 & 0 & + & + \\ 0 & 0 & + & - \end{bmatrix}\otimes\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}\times\begin{bmatrix} + & 0 & 0 & 0 \\ 0 & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & 0 & 0 & + \end{bmatrix}\otimes\begin{bmatrix} + & + \\ + & - \end{bmatrix} \quad (6)$$

and then

$$\begin{bmatrix} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{bmatrix}=\begin{bmatrix} + & 0 & 0 & 0 & + & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 & + & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & + & 0 & 0 & 0 & + \\ + & 0 & 0 & 0 & - & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 & - & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 & - & 0 \\ 0 & 0 & 0 & + & 0 & 0 & 0 & - \end{bmatrix}\times$$

$$\begin{bmatrix} + & 0 & + & 0 & & & & \\ 0 & + & 0 & + & & 0 & & \\ + & 0 & - & 0 & & & & \\ 0 & + & 0 & - & & & & \\ & & & & + & 0 & + & 0 \\ & 0 & & & 0 & + & 0 & + \\ & & & & + & 0 & - & 0 \\ & & & & 0 & + & 0 & - \end{bmatrix}\times\begin{bmatrix} + & + & 0 & 0 & & & & \\ + & - & 0 & 0 & & 0 & & \\ 0 & 0 & + & + & & & & \\ 0 & 0 & + & - & & & & \\ & & & & + & + & 0 & 0 \\ & 0 & & & + & - & 0 & 0 \\ & & & & 0 & 0 & + & + \\ & & & & 0 & 0 & + & - \end{bmatrix} \quad (7)$$

$$[H]_{16}=[H]_2\otimes[I]_2^{[2]}\otimes[I]_2\times[I]_2\otimes[H]_8=[H]_2\otimes[I]_2^{[2]}\otimes[I]_2\times[I]_2\otimes[H]_2\otimes[I]_2^{[2]}\times[I]_2^{[2]}\otimes[H]_2\otimes[I]_2\times[I]_2^{[2]}\otimes[I]_2\otimes[H]_2 \quad (8)$$

where $[I]_n$ is the identity matrix of order (n×n), which leads directly to the fast Algorithm.

and

$$[H]_{16}=\begin{bmatrix} + & + \\ + & - \end{bmatrix}\otimes\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}^{[2]}\otimes\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}\times\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}\otimes\begin{bmatrix} + & + \\ + & - \end{bmatrix}\otimes\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}^{[2]}$$

$$\times\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}^{[2]}\otimes\begin{bmatrix} + & + \\ + & - \end{bmatrix}\otimes\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}\times\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}^{[2]}\otimes\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}\otimes\begin{bmatrix} + & + \\ + & - \end{bmatrix} \quad (9)$$

$$[H]_8\otimes[H]_2=\begin{bmatrix}
+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + \\
+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -
\end{bmatrix}\times$$

$$
\begin{bmatrix}
\begin{array}{cccccccc}
+&0&0&0&+&0&0&0\\
0&+&0&0&0&+&0&0\\
0&0&+&0&0&0&+&0\\
0&0&0&+&0&0&0&+\\
+&0&0&0&-&0&0&0\\
0&+&0&0&0&-&0&0\\
0&0&+&0&0&0&-&0\\
0&0&0&+&0&0&0&-
\end{array} & 0 \\
0 &
\begin{array}{cccccccc}
+&0&0&0&+&0&0&0\\
0&+&0&0&0&+&0&0\\
0&0&+&0&0&0&+&0\\
0&0&0&+&0&0&0&+\\
+&0&0&0&-&0&0&0\\
0&+&0&0&0&-&0&0\\
0&0&+&0&0&0&-&0\\
0&0&0&+&0&0&0&-
\end{array}
\end{bmatrix}
\times
\begin{bmatrix}
\begin{array}{cccccccc}
+&0&+&0&0&0&0&0\\
0&+&0&+&0&0&0&0\\
+&0&-&0&0&0&0&0\\
0&+&0&-&+&0&0&0\\
0&0&0&0&+&0&+&0\\
0&0&0&0&0&+&0&+\\
0&0&0&0&+&0&-&0\\
0&0&0&0&0&+&0&-
\end{array} & 0 \\
0 &
\begin{array}{cccccccc}
+&0&+&0&0&0&0&0\\
0&+&0&+&0&0&0&0\\
+&0&-&0&0&0&0&0\\
0&+&0&-&0&0&0&0\\
0&0&0&0&+&0&+&0\\
0&0&0&0&0&+&0&+\\
0&0&0&0&+&0&-&0\\
0&0&0&0&0&+&0&-
\end{array}
\end{bmatrix}
\times
\begin{bmatrix}
\begin{array}{cccccccc}
+&+&0&0&0&0&0&0\\
+&-&0&0&0&0&0&0\\
0&0&+&+&0&0&0&0\\
0&0&+&-&0&0&0&0\\
0&0&0&0&+&+&0&0\\
0&0&0&0&+&-&0&0\\
0&0&0&0&0&0&+&+\\
0&0&0&0&0&0&+&-
\end{array} & 0 \\
0 &
\begin{array}{cccccccc}
+&+&0&0&0&0&0&0\\
+&-&0&0&0&0&0&0\\
0&0&+&+&0&0&0&0\\
0&0&+&-&0&0&0&0\\
0&0&0&0&+&+&0&0\\
0&0&0&0&+&-&0&0\\
0&0&0&0&0&0&+&+\\
0&0&0&0&0&0&+&-
\end{array}
\end{bmatrix}
\quad (10)
$$

Then, letting $[H]_N$ represent the sparse matrix order $N(n=\log_2 N)$, the recursive relationship is given by the expression:

$$
[H]_N = [H]_2 \otimes \underbrace{[I]_2 \otimes [I]_2 \otimes \cdots \otimes [I]_2}_{\log_2^{N-1}} \times [I]_2 \otimes [H]_{N/2} \quad (11)
$$
$$
= ([H]_2 \otimes [I]_{N/2})([I]_2 \otimes [H]_{N/2})
$$

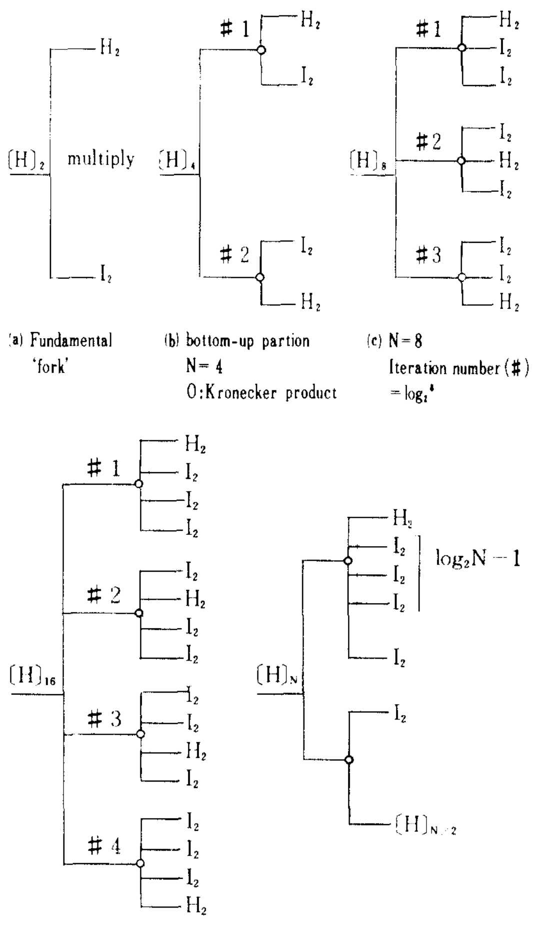
The proof of (11) is very simple. Using the algebra of Kronecker product we have

$$
[H]_N = [H]_2 \otimes [H]_{N/2} \quad (12)
$$

$$
[H]_N = ([H]_2 \otimes [I]_{N/2})([I]_2 \otimes [H]_{N/2})
$$
$$
= ([H]_2 [I]_2) \otimes ([I]_{N/2} [H]_{N/2})
$$
$$
= [H]_2 \otimes [H]_{N/2} \quad (13)
$$

From (12), the right hand side of (13) is just $[H]_N$ and the proof is complete.

The proposed method is based on matrix decomposition and 'fork' form in Fig. 1.



(a) Fundamental 'fork' (b) bottom-up partion N=4 O:Kronecker product (c) N=8 Iteration number (#) = $\log_2^N$

(d) N=16 (e) general form

Fig. 1. Sparse matrix of FHT by kronecker product.

The procedure getting the sparse matrix of the FHT can now be summarized as follows;
Step1: Search the lower order of the Hadamard sparse matrix $[H]_2$, $[I]_2$ and iteration number (#).
Step 2: Multiply the Kronecker product by the lower Hadamard Sparse matrix, according to the interation number.
Step 3: In general, multiply Kronecker product $[H]_2$, $[I]_{N/2}$ by $[I]_2$, $[H]_{n/2}$ according to iteration number.

The simple recursive relationship in (11) can now be used to formulate a sparse-matrix decomposition of $[H]_N$. Expanding the second term in (11) with successively lower orders of the Hadamard matrix results in

$$[H]_N = \pi_{i=1}^{k}([I]_i \otimes [H]_2 \otimes [I]_{N/2^i}). \tag{14}$$

Each matrix in the product form of (14) is sparse, in the sense that the i-th matrix has only two non-zero elements, +1 or −1, in each row and column. Thus the transform operation requires kN operations. For illustration $[H]_8$ is depicted below according to the decomposition in (14).

$$[H]_8 \triangleq \begin{bmatrix} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{bmatrix} = \begin{bmatrix} + & 0 & 0 & 0 & + & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 & + & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & + & 0 & 0 & 0 & + \\ + & 0 & 0 & 0 & - & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 & - & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 & - & 0 \\ 0 & 0 & 0 & + & 0 & 0 & 0 & - \end{bmatrix}$$

$$\begin{bmatrix} \begin{matrix} + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & - & 0 \\ 0 & + & 0 & - \end{matrix} & 0 \\ 0 & \begin{matrix} + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & - & 0 \\ 0 & + & 0 & - \end{matrix} \end{bmatrix} \begin{bmatrix} \begin{matrix} + & + & 0 & 0 \\ + & - & 0 & 0 \\ 0 & 0 & + & + \\ 0 & 0 & + & - \end{matrix} & 0 \\ 0 & \begin{matrix} + & + & 0 & 0 \\ + & - & 0 & 0 \\ 0 & 0 & + & + \\ 0 & 0 & + & - \end{matrix} \end{bmatrix}$$

III. Conclusions

We have presented a simple method of developing a fast Hadamard transform algorithm.

Most of the papers published to date, use a sparse matrix of basic $[H]_4$ decomposition, but we showed here lower order of Hadamard $[H]_2$ and then have represented them in general form.

This methods is presented for its simplicity and the clarity with which its decomposes a Hadamard matrix in terms of sparse matrices using Kronecker product.

References

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