⟨Research⟩

Journal of the Korean
Statistical Society
Vol. 16, No. 2, 1987.

Admissibility of Some Stepwise Bayes Estimators+

Byung Hwee Kim*

ABSTRACT

This paper treats the problem of estimating an arbitrary parametric function in the case when the parameter and sample spaces are countable and the decision space is arbitrary. Using the notions of a stepwise Bayesian procedure and finite admissibility, a theorem is proved. It shows that under some assumptions, every finitely admissible estimator is unique stepwise Bayes with respect to a finite or countable sequence of mutually orthogonal priors with finite supports. Under an additional assumption, it is shown that the converse is true as well. The first can be also extended to the case when the parameter and sample spaces are arbitrary, i.e., not necessarily countable, and the underlying probability distributions are discrete.

1. Introduction

Consider the statistical estimation problem involving the countable parameter space $\Theta\subset R^p$, the p-dimensional Euclidean space; the decision space $D\subset R^r$; the nonnegative loss function $L(\cdot,\cdot)$ on $\Theta\times D$; a random variable X which takes on values in some countable sample space $\chi\subset R^n$; and family, $\{p(\cdot\,;\theta):\theta\in\Theta\}$, of possible probability functions for X.

Suppose that it is desired to estimate some parametric function $g(\theta)$ and assume that $g(\Theta)\subseteq D$, where $g(\Theta)=\{g(\theta):\theta\in\Theta\}$. Let δ denote an estimator (possible randomized)

* Department of Mathematics, Hanyang University, Seoul 133, Korea.

+ Research supported by a grant from Hanyang University.
Part of this article was presented at the 4th Korea and Japan Conference of Statistics, July 10~12, 1986 in Seoul, Korea.

from χ to D^*, the space of all probability distribtions on D, with the risk function

$$R(\theta,\delta)=\sum_{x\in\chi}\int L(\theta, T)d\delta(x)p(x\,;\theta),$$

where for each given $x\in\chi$, T is a random variable with probability distribution $\delta(x)$ over D. This leads to the usual definitions; for example, δ^* is admissible in D^* if there does not exist any other estimator δ such that $R(\theta,\delta)\leqslant R(\theta,\delta^*)$ foɹ all $\theta\in\Theta$ with strict inequality for at least one $\theta\in\Theta$.

The (ordinary) Bayesian procedure that uses a prior distribution Π on Θ to obtain a unique Bayes estimator has been widely used as a tool to obtain admissible estimators; for example, if Θ is countable and Π gives positive probability to each $\theta\in\Theta$, then a Bayes estimator with respect to Π is admissible (see Berger, 1985). But there are some priors which yield a class of Bayes estimators rather than a unique one. Moreover, this class usually consists of inadmissible as well as admissible estimators. Hsuan (1979) proposed an idea of stepwisely applying the Bayesian procedure to extract admissible estimators out of that class, called the "Stepwise Bayesian Procedure". The idea is to use an ordered set of mutually orthogonal priors to get a stepwise Bayes estimstors. He used this idea in order to characterize the (minimal) complete class when the parameter space is finite. Later, using the same idea, Meeden and Ghosh (1981) obtained the complete class theorem in the case when the parameter and sample spaces are finite, and Brown (1981) described the complete class theorem when the sample space is finite.

But, in some cases, it is not easy to define a set of mutually orthogonal priors on an infinite parameter space. For this reason, Meeden and Ghosh (1982) introduced a notion called "finite admissibility". The basic idea of this notion is to have admissibility on certain finite subset of the parameter space. They also showed that every finitely admissible estimator is admissible, but the converse is not necessarily true.

In Section 2 we first give some assumptions imposed upon the probability function $p(x;\theta)$, the decision space D, and the loss function L. We then provide precise definitions of notions introduced above. In Section 3 we verify that, under assumptions given in Section 2, every finitely admissible estimator is unique stepwise Bayes with respect to a finite or countable sequence of mutually orthogonal priors with finite supports (Theorem 3.1), and conversely (Theorem 3.2 and Theorem 3.3). Section 4 contains an extension (Theorem 4.1) of Theorem 3.1 to the case when the sample space χ and the parameter space Θ are arbitrary, i.e., not necessarily countable, and the underlying probability distributions are discrete. Also, some closing remarks are given in this section.

2. Assumptions and definitions

Throughout this article, we make the following two assumptions:

($A1$) For each $x \in \chi$, $p(x\,;\theta) > 0$ for at least one $\theta \in \Theta$.

($A2$) D and L are such that $\sum_{\theta \in \Theta} L(\theta, d)\Pi(\theta)$ is minimized uniquely with respect to d for any prior distribution Π.

We now give the precise definitions of orthogonal priors, stepwise Bayes estimator, and finite admissibility.

Definition 2.1. Two priors Π^1 and Π^2 are said to be *orthogonal* if $\Theta(\Pi^1) \cap \Theta(\Pi^2)$ is empty where $\Theta(\Pi) = \{\theta : \Pi(\theta) > 0\}$ for any prior Π. $\Theta(\Pi)$ is called the *support* of Π.

Before giving the definition of stepwise Bayes estimator, we need the following notations:

Let $h(x\,;\Pi) = \sum_{\theta \in \Theta} p(x\,;\theta)\Pi(\theta)$ be the marginal density of X with respect to the prior Π. For a nonempty set $S = \{\Pi^i : i = 1, 2, \cdots\}$ of priors, define the following sets

$$\Lambda^1 = \{x \in \chi : h(x\,;\Pi^1) > 0\}$$

and

$$\Lambda^i = \{x \in \chi : h(x\,;\Pi^i) > 0 \text{ and } x \notin \bigcup_{i' < i} \Lambda^{i'}\} \text{ for all } i > 1.$$

Remark 2.1 Some of the Λ^i 's might be empty and that the order in which the Π^i's appear in the sequence is important in the construction of the set $\{\Lambda^i : i = 1, 2, \cdots\}$. A different ordering of the Π^i 's may result in a different set of Λ^i 's.

Definition 2.2 An estimator δ, defined on χ, is said to be *stepwise Bayes with respect to an ordered set* $\{\Pi^i : i = 1, 2, \cdots\}$ *of priors* if $\delta(x) = \delta^i(x)$ for $x \in \Lambda^i$, $i = 1, 2, \cdots$, where δ^i is Bayes with respect to Π^i. Furthermore, if $\bigcup_{i=1}^{\infty} \Lambda^i = \chi$, then such an estimator δ is said to be unique stepwise Bayes with respect to an ordered set $\{\Pi^i : i = 1, 2, \cdots\}$ of priors.

Remark 2.2 From the above definition, we notice that a stepwise Bayes estimator with respect to $\{\Pi^i\}$ is necessarily a Bayes estimator with respect to Π^1, but it need not be Bayes with respect to $\Pi^i, i = 2, 3, \cdots$. We also note that if $\{\Pi^i\}$ are such that $\bigcup_{i=1}^{\infty}$

$\Theta(\Pi^i)=\Theta$, then the stepwise Bayes estimator in this case is unique, however, the converse is not true.

Now, we give an example for Definition 2.2. This example was treated in Meeden and Ghosh (1981).

Example 2.1 Let X be a random variable with the discrete uniform distribution where the probability function of X is given by $p(x;\theta)=P_\theta(X=x)=\theta^{-1}$, $x=1,2,\cdots,\theta$; $\Theta=\{1,2,\cdots\}=\chi$. It is desired to estimate $g(\theta)=\theta$ under squared error loss and $D=[1,\infty)$. Consider a prior ditsribution Π^1 with $\Pi^1(1)=1$. Then, it is straightforward to show that $\Lambda^1=\{1\}$ and any estmators such that $\delta^1(1)=1$ are Bayes with respect to Π^1. Now, consider a second prior Π^2 with $\Pi^2(\theta)=4/[3(\theta^2-1)]$, $\theta=2,3,\cdots$. If we compute the Bayes estimators with respect to Π^2 for those x 's for which the Bayes estimators with respect to Π^1 are not defined, *i.e.*, $x=2,3,\cdots$, then we get $\delta^2(x)=2x-1$, $x=2,3,\cdots$. Note that $\Lambda^2=\{2,3,\cdots\}$. Since $\Lambda^1\cup\Lambda^2=\chi$, the estimator δ such that $\delta(x)=\delta^i(x)$, $x\in\Lambda^i$, $i=1,2$, i.e., $\delta(x)=2x-1$, $x=1,2,\cdots$, is unique stepwise Bayes with respect to $\{\Pi^i:i=1,2\}$. Note that Π^1 and Π^2 are mutually orthogonal priors such that $\Theta(\Pi^1)\cup\Theta(\Pi^2)=\Theta$ and that the estimator δ is not Bayes with respect to Π^2. It is easy to show that the estimator δ_0 such that $\delta_0(1)=3$, $\delta_0(x)=2x-1$, $x=2,3,\cdots$, is Bayes with respect to Π^2. It is also remarked that there exists no Π such that δ is the unique stepwise Bayes estimator with respect to Π. (It stands out only if more than one prior is used in a stepwise manner).

Meeden, Ghosh, and Vardeman(1985) utilized the stepwise Bayesian procedure in studying admissibility questions in nonparametric problems and in proving a detailed development of the relationship between nonparametric estimation and finite population estimation. The recent work of Brown(1984) used similar techniques to give a detailed study of the admissibility of various invariant nonparametric estimators. Brown and Farrell(1985) constructed a complete class of stepwise Bayes estimators in the setting of discrete exponential family using the idea of the stepwise Bayesian procedure. Also, Cohen and Kuo(1985), using the same idea, studied the admissibility of the empirical distribution function.

Next, we give the precise definition of finite admissibility by Meeden and Ghosh (1982).

Definition 2.3 An estimator δ is said to be *finitely admissible* if for any parameter point $\theta_0\in\Theta$ there exists a finite subset Θ_0 of Θ containing θ_0 such that when Θ_0 is taken as a restricted parameter space, δ is admissible.

Meeden and Ghosh(1982) showed that every finitely admissible estimator is admissible, but the converse is not necessarily true. Using the notions of finite admissibility and the stepwise Bayesian procedure, Meeddn and Ghosh(1983) studied admissibility in the case of choosing between experiments and gave applications to finite population sampling. Also, Ghosh and Meeden(1983), Meeden and Ghosh(1982), and Vardeman and Meeden(1983) used the same idea to give various admissibility results in finite population sampling.

3. Unique Stepwise Bayes Estmators and Finitely Admissible Estimators

Before stating and proving the main results of this section, which characterize the class of finitely admissible estimators, we need the following lemma:

Lemma 3.1 *Let Θ and χ be finite and countable, respectively. In addition to the assumptions (A1) and (A2), assume that each admissible estimator has finite risk for each $\theta \in \Theta$. Then, an estimator is admissible if and only if it is unique stepwise Bayes with respect to a finite sequence of mutually orthogonal priors.*

Since the above lemma is essentially equivalent to the Theorms 1, 2, and the related discussion in Hsuan(1979), the proof of the lemma is therefore omitted.

Now, we give a main result which shows that every finitely admissible estimator is unique stepwise Bayes.

Theorem 3.1 *Assume (A1) and (A2) and suppose that each finitely admissible estimator has finite risk for each $\theta \in \Theta$. Then, every finitely admissible estimator is unique stepwise Bayes with respect to a finite or countable sequence of mutually orthogonal priors with finite supports.*

Proof. First, note that since Θ is countable, it can be put into one-to-one correspondence with N, the set of natural numbers, and, hence, Θ can be indexed with positive integers, i.e., the elements θ of Θ can be arranged in a sequence, $\{\theta_n\}$, $n=1,2\cdots$, of distinct terms. Now, suppose that δ^* is a finitely admissible estimator. Pick $\theta_1 \in \Theta$ and a finite subset Θ_1^* of Θ containing θ_1 for which δ^* is admissible when Θ_1^* is taken as the restricted parameter space. Then, Θ_1^* yields a subset Λ_1 of χ where $\Lambda_1 = \{x \in \chi : p(x\,;\theta) > 0$ for at least one $\theta \in \Theta_1^*\}$. Note that Λ_1 is not empty by assumption (A1), and Λ_1 is at most countable. Since δ^* is admissible for the restricted problem (Θ_1^*, Λ_1)

with θ restricted to Θ_1^* and x restricted to Λ_1, and δ^* has finite risk for each $\theta\in\Theta_1^*$ in this restricted problem by assumption of the theorem, δ^* is stepwise Bayes with respect to a finite sequence, $\{\Pi_1^i\}_1^{n_1}$ say, of mutually orthogonal priors with finite supports. Note that the risk of δ^* in the restricted problem (Θ_1^*, Λ_1) is given by

$$R^{(1)}(\theta,\delta^*)=\sum_{x\in\Lambda_1}\int_D L(\theta, T)d\delta^*(x) \ \frac{p(x\,;\,\theta)}{\sum_{x\in\Lambda_1}p(x\,;\,\theta)},\quad \theta\in\Theta_1^*$$

for each given $x\in\Lambda_1$,

where T is a random variable with probability distribution $\delta^*(x)$ over D. Also note that δ^* is uniquely determined for $x\in\Lambda_1$ by assumption $(A2)$, and that $\bigcup_{i=1}^{n_1}\Theta_1^*(\Pi_1^i)\subseteq\Theta_1^*$ where $\Theta_1^*(\Pi_1^i)=\{\theta\in\Theta_1^* : \Pi_1^i(\theta)>0\}$, $i=1,2,\cdots,n_1$. Now define

$$\Theta_1^{**}=\Big[\bigcup_{i=1}^{n_1}\Theta_1^*(\Pi_1^i)\Big]\cup\Theta^1, \tag{3.1}$$

where $\Theta^1=\{\theta\in\Theta-\bigcup_{i=1}^{n_1}\Theta_1^*(\Pi_1^i) : \sum_{x\in\Lambda_1}p(x\,;\,\theta)=1\}$. Note that Θ_1^{**} also yields Λ_1 which is generated by Θ_1^*. Next, let θ_2^* be the element with the smallest indexing integer in $\Theta-\Theta_1^{**}=\{\theta\in\Theta : \theta\notin\Theta_1^{**}\}$, and let Θ_2 be a finite subset of Θ which contains θ_2^* and for which δ^* is admissible when Θ_2 is taken as the restricted parameter space. Again, Θ_2 yields a subset χ_2 of χ where $\chi_2=\{x\in\chi : p(x\,;\,\theta)>0$ for at least one $\theta\in\Theta_2\}$. Then χ_2 is not empty by assumption $(A1)$. Note that $\Theta_1^{**}\cap\Theta_2$ need not be empty. We first show that δ^* is admissible when $\Theta_2-\Theta_1^{**}\cap\Theta_2=\Theta_2^*$ is taken as a restricted parameter space. To this end, it is sufficient to consider estimators δ such that $\delta(x)=\delta^*(x)$ for $x\in\Lambda_1$. Suppose that δ^* is not admissible for the restricted problem (Θ_2^*, Λ_2) where $\Lambda_2=\{x\in\chi : p(x\,;\,\theta\}>0$ for at least one $\theta\in\Theta_2^*\}$. Note that Λ_2 is not empty by assumption $(A1)$. Then, there exists a δ' such that for all $\theta\in\Theta_2^*$,

$$R^{(2)}(\theta,\delta')\leqslant R^{(2)}(\theta,\delta^*) \tag{3.2}$$

with strict inequality for at least one $\theta\in\Theta_2^*$, where

$$R^{(2)}(\theta,\delta)=\sum_{x\in\Lambda_2}\int_D L(\theta, T)d\delta(x) \ \frac{p(x\,;\,\theta)}{\sum_{x\in\Lambda_2}p(x\,;\,\theta)},\quad \theta\in\Theta_2^*$$

is the risk of an estimator δ in the restricted problem (Θ_2^*, Λ_2) and for each given $x\in\Lambda_2$, T is a random variable with probability distribution $\delta(x)$ over D. But for $\theta\in\Theta_1^{**}\cap\Theta_2$,

$$R'(\theta,\delta')=R'(\theta,\delta^*) \tag{3.3}$$

since $\delta'(x)=\delta^*(x)$ for $x\in\Lambda_1$, where $R'(\theta,\delta)$ denotes the risk of an estimator δ in the restricted problem (Θ_2, χ_2). Hence, if follows (3.2) and (3.3) that for $\theta \in \Theta_2$,

$$\begin{aligned} R'(\theta,\delta) &= \Big[\sum_{x\in\Lambda_2} p(x;\theta)\Big]\ R^{(2)}(\theta,\delta') + \sum_{x\in X_2-\Lambda_2}\int_D L(\theta,T)d\delta'(x)p(x;\theta) \\ &= \Big[\sum_{x\in\Lambda_2} p(x;\theta)\Big]\ R^{(2)}(\theta,\delta') + \sum_{x\in X_2-\Lambda_2}\int_D L(\theta,T)d\delta^*(x)p(x;\theta) \\ &\leqslant \Big[\sum_{x\in\Lambda_2} p(x;\theta)\Big]\ R^{(2)}(\theta,\delta^*) + \sum_{x\in X_2-\Lambda_2}\int_D L(\theta,T)d\delta^*(x)p(x;\theta) \\ &= \Big[\sum_{x\in\Lambda_2}\int_D L(\theta,T)d\delta^*(x)p(x;\theta) \\ &\qquad\qquad + \sum_{x\in X_2-\Lambda_2}\int_D L(\theta,T)d\delta^*(x)p(x;\theta) \\ &= \sum_{x\in X_2}\int_D L(\theta,T)d\delta^*(x)p(x;\theta) \\ &= R'(\theta,\delta^*) \end{aligned} \tag{3.4}$$

with strict inequality for some $\theta \in \Theta_2^* \subset \Theta_2$. But (3.4) contradicts the fact that δ^* is admissible when Θ_2 is taken as the restricted paremeter space. Hence, δ^* is admissible in the restricted problem (Θ_2^*, Λ_2). Note that Θ_2^* is finite, but Λ_2 may be countable, and $R^{(2)}(\theta,\delta^*)$, the risk of δ^* in the restricted problem (Θ_2^*, Λ_2), is finite for each $\theta \in \Theta_2^*$ by assumption of the theorem. Hence, by Lemma 3.1, δ^* is stepwise Bayes with respect to a finite sequence, $\{\Pi_2^j\}_1^{n_2}$ say, of mutually orthogonal priors with finite supports, and δ^* is uniquely determined for $x \in \Lambda_2$ by assumption $(A2)$. Also note that $\bigcup_{j=1}^{n_2} \Theta_2^*(\Pi_2^j) \subseteq \Theta_2^*$, and hence, that $\Big[\bigcup_{i=1}^{n_1} \Theta_1^*(\Pi_1^i)\Big] \cup \Big[\bigcup_{j=1}^{n_2} \Theta_2^*(\Pi_2^j)\Big] \subseteq \Theta_2^{**} \cup \Theta_2^*$, where $\Theta_1^*(\Pi_1^i) \cap \Theta_2^*(\Pi_2^j)$ is empty for all $i=1,2,\cdots,n_1$, and $j=1,2,\cdots,n_2$ and Θ_1^{**} is as in (3.1). It can be also noted that δ^* is uniquely determined for $x \in \Lambda_1 \cup \Lambda_2$ by assumption $(A2)$. Continuing the above process, if at some finite stage all the x 's are used up, then we are done, and we conclude that δ^* is unique stepwise Bayes with respect to a finite sequence, $\{\Pi^i\}_1^k$ say, of mutually orthogonal priors with finite supports. It may be noted that $\bigcup_{i=1}^{k} \Theta(\Pi^i) \subset \Theta$. On the other hand, if at every finite stage all the x 's are not used up, then we have a countable sequence, $\{\Pi_l^i\}$ say, $i=1,2,\cdots,n_l$, $l=1,2,\cdots$, of mutually orthogonal priors. Now, it is clear that

$$\bigcup_{l=1}^{\infty} \Theta_l^{**} \subseteq \Theta \tag{3.5}$$

where $\Theta_l^{**} = \Big[\bigcup_{i=1}^{n_l} \Theta_l^*(\Pi_l^i)\Big] \cup \Theta^l$ with Θ_l^* and Θ^l appropriately defined at the l^{th} stage, $l=1,2,\cdots$. We show that the equality in (3.5) must happen. To this end, suppose that

$\bigcup_{l=1}^{\infty} \Theta_l^{**} \subset \Theta$. Then, we have the element with the smallest indexing integer, θ^* say, in $\Theta - \bigcup_{l=1}^{\infty} \Theta_l^{**}$. Note that at the $(j+1)^{st}$ stage of the above process θ_j^* must be a member of $\bigcup_{l=1}^{j+1} \Theta_l^{**}$, $j=1,2,\cdots$, with $\theta_1^*=\theta_1$. This implies that we must have equality in (3.5). From this fact, it follows from assumption $(A1)$ that δ^* is unique stepwise Bayes with respect to a countable sequence, $\{\Pi^i\}_1^\infty$ say, of mutually orthogonal priors with finite supports. ▨

The following theorem shows that the converse of Theorem 3.1 is also true provided it is assumed that $\Theta = \bigcup_{i=1}^{\infty} \Theta(\Pi^i)$ in addition to the assumptions of Theorem 3.1.

Theorem 3.2 *If δ^* is unique stepwise Bayes with respect to a countable sequence, $\{\Pi^i\}_1^\infty$, of mutually orthogonal priors with finite supports, $\Theta=\bigcup_{i=1}^{\infty}\Theta(\Pi^i)$, and each finitely admissible estimator has finite risk for each $\theta\in\Theta$, then δ^* is finitely admissible in the whole problem (Θ,χ).*

Proof. Suppose not, then there exists a θ_0 such that for every finite subset of Θ containing θ_0, δ^* is inadmissible. Let i_0 be such that $\Theta(\Pi^{i_0})=\{\theta\in\Theta : \Pi^{i_0}(\theta)>0\}$ contains θ_0. This i_0 always exists and is unique since $\Theta=\bigcup_{i=1}^{\infty} \Theta(\Pi^i)$. Let $\Theta'=\bigcup_{i=1}^{i_0} \Theta(\Pi^i)$. Then, since Θ' is a finite subset of Θ containing θ_0, δ^* is inadmissible when Θ' is taken to be the restricted parameter space. But δ^* is unique stepwise Bayes with respect to a finite sequence, $\{\Pi^i\}_1^{i_0}$, of mutually orthogonal priors with finite supports. Note that Θ' yield a subset χ' of χ where $\chi'=\{x\in\chi : p(x\,;\theta)>0$ for at least one $\theta\in\Theta'\}$. Also note that χ' may be at most countable. Since each finitely admissible estimator has finite risk for each $\theta\in\Theta'$ in the whole problem, it follows from Lemma 3.1 that δ^* is admissible in the restricted problem (Θ',χ'). Hence, we have a contradiction, and the proof is completed. ▨

Remark 3.1 In Theorem 3.2, condition $\Theta=\bigcup_{i=1}^{\infty} \Theta(\Pi^i)$ is needed since otherwise there may not exist any $\Theta(\Pi^i)$ containing θ_0, where θ_0 is as in the proof of the theorem. It may be also remarked that combining Theorem 3.1 with Theorem 3.2 provides a characterization of the class of all finitely admissible estimators.

In Theorem 3.2, we assumed that each finitely admissible estimator has finite risk for each $\theta\in\Theta$. But, as will be seen in the following theorem, this assumption may be replaced by assumption that for each $\theta\in\Theta$, $p(x;\theta)>0$ for only finitely many x.

Theorem 3.3 *If δ^* is unique stepwise Bayes with respect to a countable sequence, $\{\Pi^i\}_1^\infty$, of mutually orthogonal priors with finite supports and for each $\theta\in\Theta$, $p(x;\theta)>0$ for only finitely many x and $\Theta=\bigcup_{i=1}^{\infty}\Theta(\Pi^i)$, then δ^* is finitely admissible.*

Proof. With Θ' and χ' as in the proof of Theorem 3.2, χ' is a finite subset of χ since Θ' is finite. Hence, by Theorem 1 of Meeden and Ghosh (1981), δ^* is admissible in the restricted problem (Θ',χ'). ▨

Remark 3.2 All the previous theorems do not provide the characterization of the class of all admissible estimators. They only give the characterization of the class of all finitely admissible estimators. In general, it is very hard to characterize the class of all admissible estimaters in the case that the parameter space is not compact.

4. An Extension and Closing Remarks

Theorem 3.1 can be also extended to the case when the parameter and sample spaces are arbitrary subsets of the Euclidean space and the underlying probability distributions are discrete. The extension is as follows:

Theorem 4.1 *Assume (A1) and (A2) of Section 2 and assume that each finitely admissible estimator has finite risk for each $\theta\in\Theta$. Then every finitely admissible estimator is a unique stepwise Bayes estimator with respect to an ordered set $\{\Pi^\alpha:\alpha\in I\}$ of mutually orthgonal priors with finite supports where I is a well ordered set with smallest element $\alpha(1)$.*

The proof follows exactly the same lines as that of Theorem 3.1 if we apply the principle of transfinite induction.

But extensions of Theorems 3.2 and 3.3 are not available to this situation. The availability of these extensions is closely related to a question whether if δ^* is inadmissible for a restricted problem (Θ',χ'), then δ^* is inadmissible for the original problem (Θ,χ), where $\Theta'\subset\Theta$ and $\chi'=\{x\in\chi:p(x;\theta)>0$ for at least one $\theta\in\Theta'\}$. Moreover, this question is closely related to another interesting question whether if δ^* is admissible for the original problem (Θ,χ), then for any $\theta_0\in\Theta$, there exists the smallest subset Θ_0 of Θ containing θ_0 for which δ^* is admissible when Θ_0 is taken as the restricted parameter space. The second question seems to be much harder to answer than the first one. The following example shows that even the first question is not necessarily true.

Example 4.1 Let X have the probability function $p(x\,;\,\theta)=P_\theta(X=x)=\theta^{-1}$, $x=1,2,\cdots,\theta$; $\Theta=\{1,2,\cdots\}=\chi$. In this case Blyth (1974) has proved the admissibility of $\delta^*(X)=X$ as an estimator of θ under squared error loss by using a Cramér-Rao type inequality (See also Meeden and Gosh (1981)). This estimator has risk $R(\theta,\delta^*)=\theta^{-1}\sum_{x=1}^{\theta}(x-\theta)^2$. Consider the restricted problem (Θ',χ') where $\Theta'=\{2,3,\cdots\}$ and $\chi'=\{x\in\chi : p(x\,;\,\theta)>0$ for at least one $\theta\in\Theta'\}=\chi$. Also, consider an estimator δ' such that $\delta'(1)=2$, $\delta'(x)=x$, $x=2,3,\cdots$. Then, the risk of δ' is given by $R(\theta,\delta')=\theta^{-1}\{(2-\theta)^2+\sum_{x=2}^{\theta}(x-\theta)^2\}$. Now, $R(\theta,\delta^*)-R(\theta,\delta')=\theta^{-1}(2\theta-3)>0$ for all $\theta\in\Theta'$. Hence, δ^* is inadmissible for the restricted problem (Θ',χ').

References

(1) Berger, J.O. (1985). *Statistical Decision Theory and Bayesian Analysis*, 2nd Ed., Springer-Verlag, New York.

(2) Blyth, C.R. (1974). Necessary and Sufficient Conditions for Inequalities of Cramér-Rao Type, *The Annals of Statistics*, Vol.2, 464~473.

(3) Brown, L.D. (1981). A complete class Theorem for Statistical Problems with Finite Sample Spaces, *The Annals of Statistcs*, Vol.9, 1289~1300.

(4) Brown, L.D. (1984). Admissibility in Discrete and Continuous Invariant Nonparametric Estimation Problems, and in Their Multinomial Analogues, *Submitted to The Annals of Statistics.*

(5) Brown, L.D. and Farrell, R.H. (1985). Complete Class Theorems for Estimation of Multivariate Poisson Means and Related Problems, *The Annals of Statistics*, Vol.13, 706~726.

(6) Cohen, M.P. and Kuo, L. (1985). The Admissibility of the Empirical Distribution Function, *The Annals of Statistics*, Vol.13, 262~271.

(7) Ghosh, M. and Meeden, G. (1983). Estimation of the Variance in Finite Population Sampling, *Sankhya* 45, *Series* B, 362~375.

(8) Hsuan, F. (1979). A stepwise Bayesian Procedure, *The Annals of Statistics*, Vol.7, 860~868.

(9) Meeden, G. and Ghosh, M. (1981). Admissibility in Finite Problems, *The Annals of Statistics*, Vol.9, 846~852.

(10) Meeden, G. and Ghosh, M. (1982). On the Admissibility and Uniform Admissibility of Ratio Type Estimates, To appear in *the Proceedings of the Golden Jubilee Conference of the Indian Statistical Institute.*

(11) Meeden, G. and Ghosh, M. (1983). Choosing between Experiments: Applications to Finite

Population Sampling, *The Annals of Statistics*, Vol. 11, 296~305.

(12) Meeden, G., Ghosh, M., and Vardeman, S. (1985). Some Admissible Nonparametric and Related Finite Population Sampling Estimators, *The Annals of Statistics*, Vol. 13, 811~817.

(13) Vardeman, S. and Meeden, G. (1983). Admissible Estimators in Finite Population Sampling Employing Various Types of Prior Information, *Journal of Statistical Planning and Inference*, Vol. 7, 329~341.