⟨Research Paper⟩

Journal of the Korean
Statistical Society
Vol.16, No. 1, 1987

A Modification of the Combined Estimator of Inter- and Intra-Block Estimators under an Arbitrary Convex Loss Function

Youngjo Lee*

ABSTRACT

The combined estimator of inter- and intra-block estimators in incomplete block designs can be expressed as a weighted average of two location estimators. The weight should be between 0 and 1. However, the negative variance component estimate could result in the weight being negative or larger than 1. In this paper, we show that if two location estimators have symmetric unimodal distributions, truncating the weight to 0 or 1 accordingly improves the combined estimator under an arbitrary convex loss function.

1. Main Results

It has been known that the combined estimator of inter- and intrablock estimators can be expressed as a weighted average of two location estimators (for example, see Khatri and Shah, 1975). Bhattacharaya (1983) pointed out that the weight could become negative in many experimental situations.

For a moment, consider a simple location problem. Suppose we have a random variable Y with density function $f_\theta(y)$ with mean θ. Then, the density function f_θ will be called symmetric if $f_\theta(y)=f(|y-\theta|)$ and unimodal if f is nonincreasing function of $|y-\theta|$.

Consider estimators of the form:

$$\hat{\theta}=\phi(|Y|)Y$$

and

$$\hat{\theta}_T=\max\{0, \phi(|Y|)\}Y.$$

* Department of Statistics, Hallym University, Chunchon, KOREA

For the Stein estimator, Stein (1966) and Lehmann (1983, page 302) showed that $\hat{\theta}_T$ is better than $\hat{\theta}$ under the quadratic loss when the distribution of Y is normal. Berger and Bock (1976) extended the above result to the symmetric unimodal distribution. This result could be extended to an arbitrary convex loss. We will consider the symmetric loss function only, since it measures the distance between the estimator and the true parameter value.

Theorem 1. If Y has a symmetric unimodal distribution, then

$$R(\hat{\theta},\ \theta) \geq R(\hat{\theta}_T, \theta) \text{ for all } \theta \in R,$$

where $R(\hat{\theta}, \theta) = E\ L(\hat{\theta}-\theta)$ and L is an arbitrary convex loss function.

Proof. See the Appendix. ▨

Theorem 1 could be extended easily to the multivariate location estimator using the co-ordinatewise concepts used in Berger and Bock's 1976 paper. These concepts concern the properties of each co-ordinate holding other co-orinates fixed. For example, the co-ordinatewise symmetric distribution means that the density function of one coordinate is symmetric holding other co-ordinates fixed.

Let $Z = Y - c$ and $\theta^* = \theta - c$ for some constant c. Then Z has a symmetric unimodal distribution with mean θ^*. From Theorem 1, it can be shown that

$$R(\phi(|Z|)Z+c,\ \theta) \geq R\{\max\{0, \phi(|Z|)\}Z+c, \theta) \text{ for all } \theta \in R,$$

since $L(\phi(|Z|)Z-\theta^*) = L(\phi(|Z|)Z+c-\theta)$.

Suppose we have another random variable X which has a symmetric unimodal distribution with mean θ and is independent of Y. The distribution of X is not necessarily the same as that of Y. Using the same arguments conditioning on X, we can show that

$$R(\hat{\theta}^c,\ \theta) \geq R(\hat{\theta}^c_0,\ \theta) \text{ for all } \theta \in R,$$

where $\hat{\theta}^c = \phi(|Y-X|)(Y-X)+X$

and

$$\hat{\theta}^c_0 = \max\{0, \phi(|Y-X|)\}(Y-X)+X.$$

For simplicity we will use ϕ instead of $\phi(|Y-X|)$. Note that

$$\hat{\theta}^c = (1-\phi)(X-Y)+Y$$

and

$$\hat{\theta}^c_1 = \max\{0,\ 1-\phi\}(X-Y)+Y$$
$$= \min\{1, \phi\}(Y-X)+X.$$

Since our arguments are symmetric with respect to X and Y, we can show that

$$R(\hat{\theta}^c,\ \theta) \geq R(\hat{\theta}^c_1,\ \theta) \text{ for all } \theta \in R$$

by applying the same arguments conditioning on Y.

Theorem 2. When X and Y have symmetric unimodal distributions with mean θ, then under an arbitrary convex loss, we have

$$R(\hat{\theta}^c,\ \theta) \geq R(\hat{\theta}^c_T,\ \theta) \text{ for all } \theta \in R,$$

where $\hat{\theta}^c_T = \min\{1,\ \max\{0,\ \phi\}\}(Y-X)+X$.

Proof. First show $R(\hat{\theta}^c,\ \theta) \geq R(\hat{\theta}^c_0,\ \theta)$ conditioning on X. Then show $R(\hat{\theta}^c_0,\ \theta) \geq R(\hat{\theta}^c_T,\ \theta)$ conditioning on Y. ▩

2. A Modification of the Combined Estimator for Inter- and Intra-Block Estimators.

Following the notations of Bhattacharaya(1983), we can express the recovery of inter- and intra-block information as follows: Suppose we have independent random variables X, Y, S, T and W_i, $i=1,2,\cdots, q$ such that

$$X \sim N(\theta,\ \sigma^2_X),$$
$$Y \sim N(\theta,\ \sigma^2_Y),$$
$$S/\sigma^2_X \sim \chi^2_m,$$
$$T/\sigma^2_Y \sim \chi^2_n,$$

and

$$W_i/(\alpha_i\sigma^2_X+\beta_i\sigma^2_Y) \sim \chi^2_1 \text{ for } i=1,2,\cdots,q,$$

where α_i's and β_i's are known constants and σ^2_X and σ^2_Y are unknown parameters. Let $W_0=(Y-X)^2$. Interpretations of X, Y, S, T and W_i's are as follows:

X and Y are intra-block and inter-block estimators of a given cannonical contrast which is estimable form both intra-block and inter-block analysis. W_i's are squared differences between intra-block and inter-block estimators of other cannonical contrasts. S and T are intra-block and inter-block sum of squares.

Then, the combined estimator of intra- and inter-block estimators is

$$\hat{\theta}^c = \phi(Y-X)+X,$$

where ϕ is a measurable functions of $S, T, W_i, i=0,1,2,\cdots,q$. Since our arguments in the previous section do not depend on the values of σ^2_X and σ^2_Y, using the same arguments conditioning on $S, T, W_0, W_1, \cdots, W_q$, we can show that under an arbitrary convex loss function,

$$R(\hat{\theta}^c,\ \theta) \geq R(\hat{\theta}^c_T,\ \theta) \tag{2.1}$$

for all values of θ, σ_X^2 and σ_Y^2,

where $\hat{\theta}_T^c = \min\{1, \max\{0, \phi\}\}(Y-X)+X$ and ϕ is as above.

Note that S, T and W_i's are even location free statistics and therefore, ϕ is also. $\hat{\theta}^c$ and $\hat{\theta}_T^c$ are odd location statistics. In this case, Seely and Hogg(1982) showed that $\hat{\theta}^c$ and $\hat{\theta}_T^c$ are unbiased if within and between block error distributions are symmetric. So the risk under the quadratic loss function is a multiple of variance. Shah(1971) and Bhattacharaya(1983) investigated the variance of truncated estimator when the lower limit of variance components ratio is known.

The proof in the previous section does not require any specific distribution for X, Y, S, T and W_i's but does require independence of these random variable. These random variables are obtained by linear transformations and are uncorrelated. However, the normality assumption is necessary to hold independence of these random variables. Therefore, it might be interesting to prove the inequality (2.1) using properties of even location free statistics and odd location statistics and the uncorrelatedness of X, Y, S, T and W_i's without assuming independence.

Appendix

Proof of Theorem.

Without loss of generality, assume $\theta \geq 0$. Then,

$$R(\hat{\theta}, \theta) - R(\hat{\theta}_T, \theta)$$
$$= \int_{\phi<0} \{L(\hat{\theta}-\theta) - L(\theta)\} f_\theta(y)\, dy.$$

By transforming y to $-Z$ when $y<0$ and then transforming Z to y, the equation above becomes

$$\int_{\substack{\phi<0 \\ y>0}} \{L(\phi y-\theta) - L(\theta)\} f_\theta(y)\, dy$$
$$+\int_{\substack{\phi<0 \\ y>0}} \{L(-\phi y-\theta) - L(\theta)\} f_\theta(-y)\, dy$$
$$\geq \int_{\substack{\phi<0 \\ y>0}} \{L(\phi y-\theta) - L(\theta)\} (f_\theta(y) - f_\theta(-y))\, dy,$$

since $L(-\phi y-\theta) - L(\theta) \geq -L(\phi y-\theta) + L(\theta)$ by the convexity of L. Note that since $\theta \geq 0$,

$$L(\phi y-\theta) \geq L(\theta) \text{ when } \phi y<0$$

and

$$f_\theta(y) \geq f_\theta(-y) \text{ when } y>0.$$

Therefore, we have the desired result. ▩

References

(1) Berger, J.O. and Bock, M.E.(1976). Eliminating Singularities of Stein-type Estimators of Location Vectors, *Journal of the the Royal Statistical Society*, B, No. 2, 166~170.

(2) Bhattacharaya, C.G.(1983). Use of Modified Estimator in Recovery of Inter-block Information. *Sankhyā, B*, Vol. 45, 466~470.

(3) Khatri, C.G. and Shah, K.R.(1975). Exact Variance of Combined Inter- and Intra-Block Estimators in Incomplete Block Desings, *Journal of The American Statistical Association*, Vol. 70, 402~406.

(4) Lehmann, E.L.(1983), *Theory of Point Estimation*, Wiley, New York.

(5) Seely, Justus aud Hogg, R.V.(1982), Symmetrically Distributed and Unbiased Estimators in Linear Models, *Communications in Statistics*, A, Vol. 11, 721~729.

(6) Shah, K.R.(1971), Use of Truncated Estimator of Variance Ratio in Recovery of Inter-Block Infermation, *The Annals of Mathematical Statistics*, Vol. 42, 816~819.

(7) Stein, C.(1966), An Approach to the Recovery of Inter-Block Information in Balanced Incomplete Block Design, *Research Papers in Statistics*, ed., F.N. David, Wiley, New York.