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On the Improved Green Integral Equation applied to the Water-wave Radiation-Diffraction Problem

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Abstract

It is shown that irregular frequencies in the source and normal doublet distribution method, can be eliminated if the Green function associated with Kelvin's source of pulsating strength, is modified only in the region inside the body at the level of the undisturbed free surface. The system of the resulting Green integral equation is augmented without loss of the square-integrable property of its kernel so that the discretisation yields N linearly independent equations for N unknown variables.

From the solution, the potential and velocity at any point on the wetted surface of a surface-piercing body can be found using the properties of the double layer composed of the source and normal doublet distribution.

1. Introduction

The source and normal doublet distribution method as well as the source distribution method are in common use to solve a linearized water-wave radiation-diffraction problem. The source integral equation makes use of the normal velocity jump across the simple layer where sources are distributed while the Green integral equation makes use of the potential jump across the double layer where sources and normal doublets are distributed. These two methods employ the Green function associated with Kelvin's source of pulsating strength[1, 2].

Although the solution of the original boundary-value problem always exist, the solution of integral equations cannot be found at so-called irregular frequencies when the oscillating body surface intersects the free surface. More precisely, the solution of source integral equation does not exist while the solution of Green integral equation becomes undetermined at irregular frequencies[3, 4].

In this paper, the Green integral equation is improved so as to provide correct solutions for all frequencies including irregular frequencies.

2. Review of the Green integral equation formulation

The fluid is assumed to occupy a space D bounded by the wetted surface S of a rigid body and by the free surface F of deep water under gravity. The free surface is assumed to extend in all directions. The body performs simple harmonic oscillations of small amplitude with circular frequency ω. Cartesian coordinates (x, y, z) are employed with the origin 0 in the horizontal plane inside the body at the level of the undisturbed free surface F and the z axis vertically upwards.

With the usual assumptions of an inviscid incompressible fluid and irrotational flow without capillarity, the fluid velocity $\vec{v}$ is given by the gradient of a velocity potential Re $\{f(x, y, z)e^{-i\omega t}\}$, where $f(x, y, z)$ is a complex-valued spatial function. The governing

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equations of this well-known linear boundary-value problem are given below, where the time factor $e^{-i\omega t}$ is eliminated by virtue of the linearization:

$$\nabla^2 f(x,y,z)=0 \quad \text{in } D \tag{1}$$

$$-Kf+\frac{\partial f}{\partial z}=0 \quad \text{on } F \tag{2}$$

$$\frac{\partial f(P)}{\partial n}=Vn(P) \quad \text{on } S \tag{3}$$

$$\lim_{z\to-\infty}\frac{\partial f}{\partial z}=0 \tag{4}$$

$$\lim_{R\to\infty}\sqrt{R}\left(\frac{\partial f}{\partial R}-iKf\right)=0,\ R=\sqrt{x^2+y^2} \tag{5}$$

where $K=\frac{\omega^2}{g}$ is the wave number, $Vn\ (P)$ is a prescribed real or complex-valued function of position and $\frac{\partial}{\partial n}$ denotes differentiation along the normal from S into D.

We employ the Green function $G(P,M;K)$ where M denotes the source point and P the field point. It satisfies all of the above-mentioned governing equations except the body boundary condition(3). Thus the source is Kelvin's source of pulsating strength. In this paper, the classical expression of such a function is taken[5]:

$$G(P,M;K)=\frac{-1}{4\pi}\left[\frac{1}{|PM|}+\frac{1}{|PM_1|}+H(P,M;K)\right] \tag{6}$$

$$|PM|=\sqrt{(x_P-x_M)^2+(y_P-y_M)^2+(z_P-z_M)^2}$$

$$|PM_1|=\sqrt{(x_P-x_M)^2+(y_P-y_M)_2+(z_P+z_M)^2}$$

where $H(P,M;K)$ is a complex-valued harmonic function regular in the lower helf-space $z_P\leq 0$ with $z_M<0$.

It should be noted that $G(P,M;K)$ satisfies the free-surface condition on the horizontal plane denoted by Fi, which lies inside the body at the level of the undisturbed free surface. The space inside the body bounded by the closed surface $Fi\cup S$ is denoted by Di. The boundary at infinity of the lower half-space is denoted by Σ.

Applying Green's theorem to the potential and the Green function in the fluid region D, we find the following integral relations:

$$\iint_S\left[\frac{\partial f(M)}{\partial n}G(P,M;K)-f(M)\frac{\partial G(P,M;K)}{\partial n_M}\right]dS_M$$

$$=\begin{cases} f(P) \text{ for } P\in D & (7)\\ \frac{1}{2}f(P) \text{ for } P\in S & (8)\\ 0 \text{ for } P\notin D\cup S & (9)\end{cases}$$

where $\frac{\partial f}{\partial n}$ and f denote respectively the density of sources and normal doublets distributed over S and $\iint$ the principal value integral.

From (8), we can obtain a Fredholm integral equation of the second kind for f:

$$\frac{1}{2}f(P)+\iint_S f(M)\frac{\partial G(P,M;K)}{\partial n_M}dS_M$$

$$=\iint_S\frac{\partial f(M)}{\partial n}G(P,M;K)dS_M,\ P\in S \tag{10}$$

It is well known that the solution of (10) becomes undetermined at irregular frequencies where the adjoint interior potential boundary value problem admits characteristic functions.

3. Construction of the Improved Green Integral Equation

The original boundary-value problem can be regarded as a Neumann-Dirichlet problem since the free surface condition (2) is a combination of Neumann and Dirichlet conditions. It is well known that the solution of a Neumann problem is quasi-unique while the solution of a Dirichlet problem is always unique. Since the equation (10) takes the form of a Neumann problem, the uniqueness of its solution cannot be achieved unless the condition (9) is satisfied.

Consider now the adjoint interior potential boundary-value problem in Di characterized by

$$\nabla^2\hat{f}=0 \text{ in } Di \tag{11}$$

$$-k\hat{f}+\frac{\partial\hat{f}}{\partial z}=0 \text{ on } Fi \tag{12}$$

$$\hat{f}=0 \text{ on } S \tag{13}$$

Here $\hat{f}$ denotes a potential defined in Di.

It can be shown that the above interior problem possesses nontrivial solutions at irregular frequencies Kirr. It signifies that the condition (9) cannot be satisfied at Kirr and the solution of (10) loses its uniqueness. To avoid this, one can replace (12) by the following one in order that the interior problem cannot admit nontrivial solutions.

$$\hat{f}(P)=0 \text{ for } P\in Fi \tag{14}$$

Now the interior problem characterized by (11),

(13) and (14) becomes a Dirichlet problem which is always uniquely solvable: in this case we have the trivial solution; i.e., $\hat{f}=0$ in Di. So, the condition (14) will be an additional one to the Green integral equation for a surface-piercing body, which assures the uniqueness of the solution. It follows that a proper combination of this additional condition and the equation (10) will provide an improved integral equation which comprises the complete formulation of the problem.

For this purpose, (14) should be replaced by equivalent integral equations given below:

$$\hat{f}(P)+\iint_S f(M)\frac{\partial G(P,M;K)}{\partial n_M}dS_M=\iint_S \frac{\partial f(M)}{\partial n}G(P,M;K)dS_M \quad \text{for } P\in Fi \tag{15}$$

$$\iint_{Fi}\hat{f}(M)\frac{\partial G(P,M;K)}{\partial n_M}dS_M=0 \quad \text{for } P\in S \tag{16}$$

It should be noted that, from (9),

$$\hat{f}(P)=-\iint_S\left[f(M)\frac{\partial G(P,M;K)}{\partial n_M}-\frac{\partial f(M)}{\partial n}G(P,M;K)\right]dS_M \quad \text{for } P\in Fi \tag{17}$$

Therefore the equation (15) together with the equation (16) comprise the condition (14).

Combining (10), (15) and (16), and substituting the condition (3), the following integral equation is found:

$$\frac{f(P)}{2}\bigg|_{P\in S}+\iint_{S\cup Fi}\phi(M)\frac{\partial G(P,M;K)}{\partial n_M}\bigg|_{P\in S}dS_M+\hat{f}(P)\bigg|_{P\in Fi}$$
$$+\iint_S f(M)\frac{\partial G(P,M;K)}{\partial n_M}\bigg|_{P\in Fi}dS_M=\iint_S V_n(M)G(P,M;K)\bigg|_{P\in S\cup Fi}dS_M \tag{18}$$

Here

$$\phi(P)=\begin{cases} f(P) & \text{for } P\in S \quad (19)\\ \hat{f}(P) & \text{for } P\in Fi \quad (20)\end{cases}$$

Making use of the following identity

$$\hat{f}(P)=\frac{\hat{f}(P)}{2}-\frac{1}{4\pi}\iint_{Fi}\hat{f}(M)\frac{\partial}{\partial n}\frac{1}{|PM|}\cdot dS_M \quad \text{for } P\in Fi \tag{21}$$

and defining that

$$G'(P,M;K)=-\frac{1}{4\pi}\left\{\frac{1}{|PM|}+\frac{1}{|PM_1|}+H(P,M;K)\cdot[1-\delta(z_P-0)\cdot\delta(z_M-0)]\right\} \tag{22}$$

the equation (19) can be arranged as given below:

$$\frac{\phi(P)}{2}+\iint_{S\cup Fi}\phi(M)\frac{\partial G'(P,M;K)}{\partial n_M}dS_M=\iint_S V_n(M)\cdot G'(P,M;K)dS_M,\quad P\in S\cup Fi \tag{23}$$

It should be noted that the equation (23), say the improved Green integral equation, with the modified Green function $G'(P,M;K)$ is merely another expression of the equation (18).

4. Calculation of potential and velocity

The solution of (23) provides the value of potential at each control point P. From this the added mass and wave damping coefficients as well as the wave exciting forces can be found.

But, in a second-order problem such as drift force calculation, the value of potential at a point different from the control point is required. In such a case, the following formulas issued from the integral relations (7) and (8) can be used:

$$f(P)=2\iint_S\left[V_n(M)G(P,M;K)-f(M)\frac{\partial G(P,M;K)}{\partial n_M}\right]dS_M \text{ for } P\in S \tag{24}$$

$$f(P)=\iint_S\left[V_n(M)G(P,M;K)-f(M)\frac{\partial G(P,M;K)}{\partial n_M}\right]dS_M \text{ for } P\in D \tag{25}$$

The formula (24) can be used to calculate the wave height at the water line of a surface-piercing body.

As for the velocity calculation on the wetted surface, the tangential velocity jump must be taken into account[6]:

$$\frac{\partial f(P)}{\partial l}\bigg|_{P\in D}-\frac{\partial f(P)}{\partial l}\bigg|_{P\in S}=\frac{1}{2}\frac{\partial f(P)}{\partial l}\bigg|_{P\in D} \tag{26}$$

Here $\hat{l}$ denotes a unit vector tangent to S.

Making use of the equation (26), the velocity on S can be found:

$$\vec{v}(P)=V_n(P)\vec{n}+2\vec{l}\iint_S\left[V_n(M)\frac{\partial G(P,M;K)}{\partial l_P}-f(M)\frac{\partial^2 G(P,M;K)}{\partial l_P \partial n_M}\right]dS_M$$
$$+2\vec{\tau}\iint_S\left[V_n(M)\frac{\partial G(P,M;K)}{\partial \tau_P}-f(M)\frac{\partial^2 G(P,M;K)}{\partial \tau_P \partial n_M}\right]dS_M \text{ for } P\in S \tag{27}$$

Here

$$\vec{n}=\vec{l}\times\vec{\tau} \text{ on } S \tag{28}$$

The velocity when the point P is in D, can be found from the following formula:

$$\vec{v}(P)=\iint_S\left[V_n(M)\cdot\vec{\nabla}_P G(P,M;K)-f(M)\vec{\nabla}_P\frac{\partial G(P,M;K)}{\partial n_M}\right] dS_M \tag{29}$$

Here $\vec{\nabla}_P$ denotes the gradient with respect to the point P.

5. Numerical results and dicussions

Discretizing S and Fi in N and NF facets respectively, we have the following NT-by-NT linear system with $NT=N+NF$:

$$\frac{\phi_i}{2}+\sum_{j=1}^{NT}\phi_j\iint_{\delta S_j}\frac{\partial G'(Pi,M;K)}{\partial n_M}\,dS_M=\sum_{j=1}^{N}V_{nj}\iint_{\delta S_j}G'(Pi,M;K)dS_M,\quad i=1,\ NT \tag{30}$$

More explicitly

$$\begin{bmatrix} A_{11} & \cdots & A_{1N} & \cdots & A_{1NT} \\ \vdots & & \vdots & & \vdots \\ A_{N1} & \cdots & A_{NN} & \cdots & A_{NNT} \\ \vdots & & 1 & 0 \cdots & 0 \\ \vdots & & 0 & \ddots & \vdots \\ A_{NT1} & \cdots & A_{NTN} & 0 \cdots 0 & 1 \end{bmatrix}\begin{bmatrix}\phi_1 \\ \vdots \\ \phi_N \\ \vdots \\ \phi_{NT}\end{bmatrix}=\begin{bmatrix} B_{11} & \cdots & B_{1N} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NN} \\ \vdots & & \vdots \\ B_{NT1} & \cdots & B_{NTN}\end{bmatrix}\begin{bmatrix}V_{n1} \\ \vdots \\ V_{nN}\end{bmatrix} \tag{31}$$

with

$$A_{ij}=\vec{n}_j\cdot\iint_{\delta S_j}\overrightarrow{\text{grad}}_M G(P_i,M;K)\ dS_M \tag{32}$$

$$B_{ij}=\iint_{\delta S_j}G(Pi,M;K)\ dS_M \tag{33}$$

The identity matrix in (31) is resulted from the

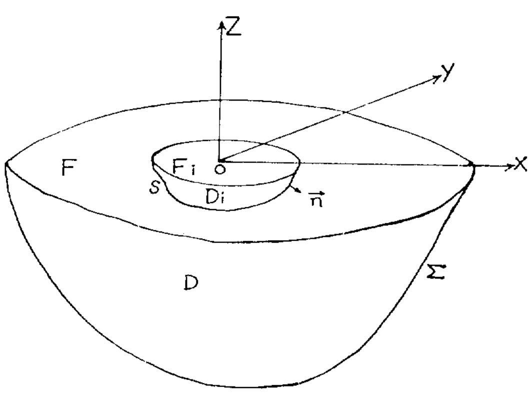


Fig. 1. a Definition of regions and boundaries

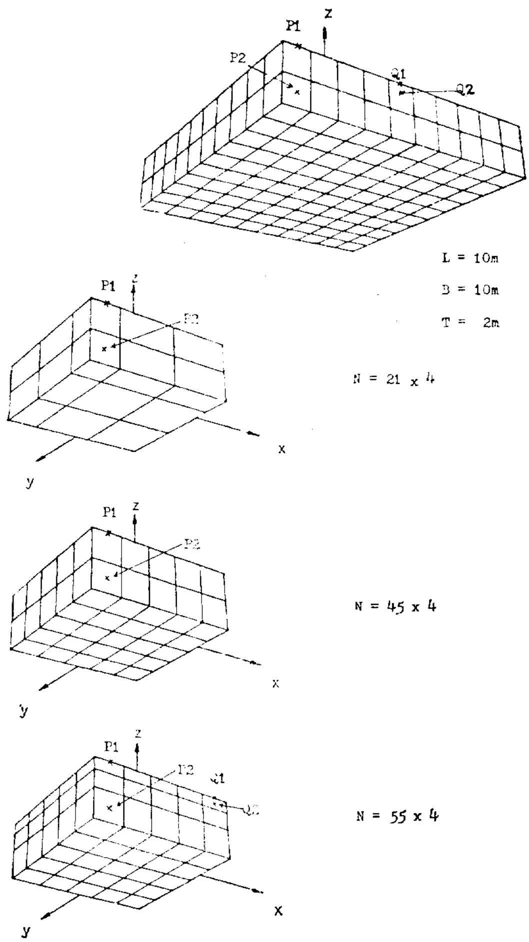


Fig. 1. b Facet representation of the box

definition of the Green function $G'(P, M; K)$ on Fi. Since G' is otherwise identical with G, there is no substantial modification of the Green function.

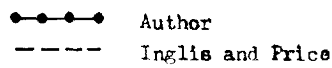
Hydrodynamic coefficients of a surface-piercing rectangular box oscillation in water of infinite depth are evaluated using the method described in this paper with the Green function represented by Guevel[7]. The box has dimensions $10\times10\times2$m and its quarter is represented by 21 to 55 facets over the wetted surface and 4 facets over the waterplane as shown in Fig. 1. b. Since the body has two planes of symmetry, the control points P lie in the body surface which belongs to one quadrant. But, the integrations in M are carried out over the entire surface $S\cup Fi$.

The added mass and damping coefficients associated with surge and heave motions are non-dimensionalized by ρL^3 and $\rho L^3\sqrt{g/L}$ where ρ is the density of fluid, L the length of the box and g the gravitational acceleration. The results are compared in Figs. 2 to 5 with the numerical results obtained from the solution of the source integral equation[8]. There is a good agreement between them when the frequencies are smaller than 2. But the results from the source integral equation become absurd as soon as the frequency approches to the first irregular frequency. It can be shown that the irregular frequencies for a rectangular box of length L, beam B and draft T is given by

$$K_{irr}=\gamma\coth\gamma T\,[\text{or }\omega_{irr}=\sqrt{g\gamma\coth\gamma T}\,] \qquad (34)$$

for

$$\gamma=\pi\sqrt{\left(\frac{k}{L}\right)^2+\left(\frac{m}{B}\right)^2},\quad k=1,2,3,\cdots \quad m=1,2,3,\cdots$$



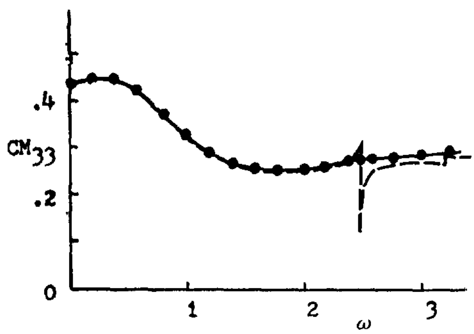


Fig. 2 Heave added mass coefficient

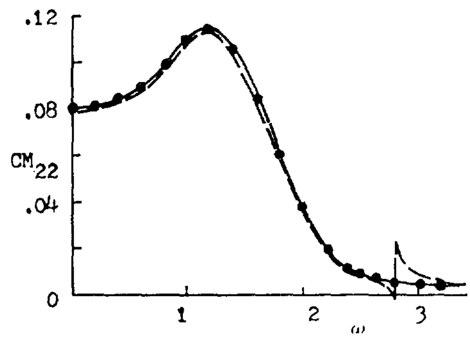


Fig. 3 Sway added mass coefficient

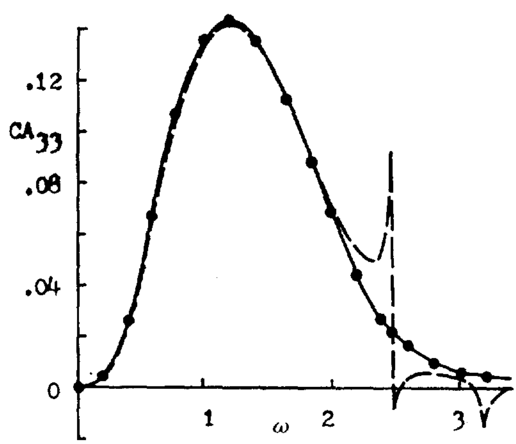


Fig. 4 Heave damping coefficient

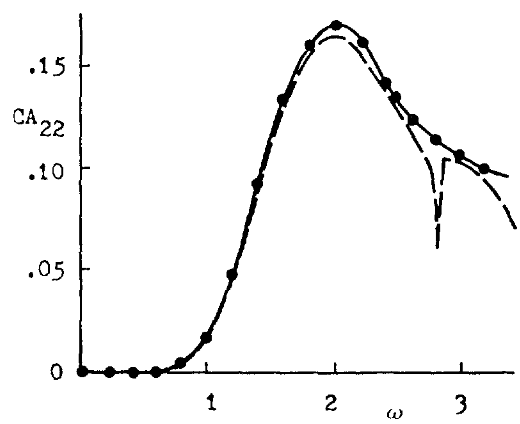


Fig. 5 Sway damping coefficient

Table 1.

ω_{irr}	CM_{22}	CM_{33}	CM_{44}	CM_{66}	CA_{22}	CA_{33}	CA_{44}	CA_{66}
2.476	0.0100	0.2754	0.0129	0.0088	0.1356	0.0223	0.00034	0.0281
2.788	0.0092	0.2842	0.0129	0.0025	0.1166	0.0096	0.00033	0.0214
3.038	0.0077	0.2888	0.0130	0.0019	0.1075	0.0058	0.00034	0.0149
3.131	0.0068	0.2910	0.0130	0.0022	0.1020	0.0046	0.00034	0.0127

CM : added mass coefficients
CA : damping coefficients

As shown in the figures, the results from the present method are free from the irregular frequencies. The added mass and damping coefficients at the first 4 irregular frequencies are again shown in the table 1.

The numerical results free from the irregular frequencies for a two-dimensional body are already presented in the earlier works of the author[9, 10]. There are also some propositions to eliminate the irregular frequencies in the Green integral equation [2, 11, 12]. Kleinman has combined (8) and (9) and derived an integral equation which appears identical with the equation (23). But the usual Green function associated with Kelvin's source is not defined when the points P and M lie simultaneously on Fi. If the usual Green function G is replaced by the modified Green function G' in the equation (23), the equation is no more identical with the equation (18). So the equation (23) is quite different from the one derived by Kleinman. Ohmatsu has combined (8) and (9) with P at the origin of axes on Fi. He has also imposed another condition which demands that the tangential velocity vanish at the origin. His numerical procedure requires the least-square orthonormalization since the domain and range of the Kernel of his integral equation do not coincide. He has also presented numerical results which are free from the first three irregular frequencies for a two-dimensional body. The method proposed by Kobus is the basis of Ohmatsu's method. The only difference is that the latter makes use of dummy unknowns to retain the square system of equations. These two methods are applied to a two-dimensional problem. It seems that the numerical results for a three-dimensional problem which are free from irregular frequencies using a method different from the present one are not yet available.

In the Table 2, the diffraction potentials and two components of tangential velocities at points P_1 and P_2 as shown in the Fig. 1, are presented for three numbers of N to show the influence of the facet size on the numerical solution.

From the table, it can be shown that 21 facets on the quarter of S are sufficient for the calculation of added mass and damping coefficients when the frequency is 1.4. However, the numerical results presented in Figs. 2 to 5 as well as in the Table 1 have been obtained from computations with 45 facets on the quarter of S and 4 facets on the quarter of Fi to give reliable results for frequencies higher than 1.4.

Table 2.

$\omega=1.4$		$N=21\times4$	$N=45\times4$	$N=55\times4$
P_1	$\tilde{f}$	$0.0849+0.304i$	$0.087+0.303i$	$0.0888+0.303i$
P_2	$\tilde{f}$	$0.0618+0.205i$	$0.0637+0.204i$	$0.0636+2.204i$
	$\tilde{v}_x$	$0.547+0.302i$	$0.565+0.295i$	$0.566+0.295i$
	$\tilde{v}_z$	$0.192+0.694i$	$0.205+0.69i$	$0.202+0.696i$

$\tilde{f}=f/a\omega L$
$\tilde{v}=v/a\omega$
a : amplitude of incident wave

Table 3.

$\omega=1.4$, $N=55\times4$		Green eq.	Source eq.
Q_1	$\tilde{f}$	$0.22+0.378i$	$0.22+0.369i$
Q_2	$\tilde{f}$	$0.214+0.365i$	$0.214+0.355i$
	$\tilde{v}_x$	$0.055+0.0466i$	$0.0493+0.026i$
	$\tilde{v}_z$	$0.431+0.786i$	$0.436+0.748i$

The numerical results from the present method are also compared with those from the source integral equation when the frequency is smaller than the first irregular frequency. The computations for the source integral equation are carried out using the usual Green function. The diffraction potentials and two components of tangential velocities at points Q_1 and Q_2 as shown in Fig. 1, are presented in the Table 3.

From the table, it can be shown that the potentials calculated by two methods are in good agreement. But the tangential velocities calculated from the Green integral equation seems to be more accurate than those from the source integral equation. The above fact can easily be verified by the numerical differentiation of the potential on S which provides also the tangential velocity.

In principle, the solution of (31) converges to the solution of (23) as N, the number of facets on the wetted surface and NF, the number of facets on the water plane tend to infinity [13]. But in practice, an appropriate number is given to N according to the dimensions of the floating body relative to the wave length. As for the number NF, it can be reduced to the number equal to the mode of irregular frequencies of (10). For example, one facet representation of the entire water plane Fi is sufficient to obtain the solution of (23) by making use of (31) when the wave number is equal to the first irregular frequency of (10). From this, it can be understood that the elimination of irregular frequencies signifies, in a sense, the removal of the non-physical sloshing modes inside the floating body[14]. Although any number greater than or equal to the mode of an irregular frequency can be given to NF to yield the same and correct solution of(31), the number equal to the mode of the irregular frequency would suffice in the computation. For a body of arbitrary shape, the prediction of irregular frequencies can be done by applying the formula (34) to the rectangular box comprising the body.

6. Conclusion

It has been shown that the improved Green integral equation is always uniquely solvable. The numerical results for a rectangular box are also presented to show that there is no irregular behaviour in the solution of the equation.

The improved Green integral equation can be used to calculate the potentials and velocities at any point on the wetted surface of a surfacepiercing body. It will be very useful to calculate second-order quantities. Moreover, the solution in time domain can easily be found from the Fourier transform of the solution in frequency domain which has no irregular frequency.

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