論 文
36～5～7

선형 다변수 시스템에 있어서 시간비중 성능지수를 이용한 최적 상수 PIDM 궤환 제어기

Optimal Constant PIDM Feedback Controller using Time Weighted Performance Index for Linear Multivariable Systems

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요 약

본 논문에서는 선형다변수 시스템에 있어서 시간 비중 성능지수를 최소화하는 비례－적분－미분 및 측정 가능한 변수를 이용한 궤환제어기의 최적화에 대하여 연구하였다. 최적에 대한 필요조건이 유도되며 최적 제어 이득을 구하는 알고리즘이 제시된다. 또한 실질적인 예를 통하여 이러한 시간 비중 성능지수를 이용하여 제어기를 설계함으로서 폐루프 시스템의 과도 응답특성을 향상시킬 수 있음을 보였다.

Abstract

The design problem of optimal constant PIDM (proportional–integral–derivative and measurable variable) feedback controller for linear time–invariannt systems is investigated with the time–weighted quadratic performance index. Necessary conditions for an optimality of the controller are derived and an algorithm for computing the optimal feedback gain is presented. It is shown via example that the design method using the time–weighted quadratic performance index improves the transient responses of the closed–loop system.

1. Introduction

Performance index is a single measure of the system's performance which emphasizes characteristics of the response that are deemed to be important. Consider a second–order system as shown Fig. 1. It is intended to find the optimal damping ratio which determines the various performance indices. A fairly useful performance index is the integral of the absolute magnitude of the error (IAE) criterion which is defined as

$$J_1=\int_0^{\infty}|e(t)|dt. \tag{1}$$

By utilizing the magnitude of the error, the integral expression increases for either positive or negative error, and results in a fairly good underdamped system. For a second–order system, this performance index has a mi-

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nimum for a damping ratio of approximately 0.7

Another usefull performance index is the integral of the squared error (ISE) criterion which is defined as

$$J_2=\int_0^\infty e^2(t)\,dt. \tag{2}$$

By focusing on the square of the error function, it penalizes both positive and negative values of the error. For a second-order system, this performance index has a minimun for a damping ratio of 0.5. This performance index has a merit that it may lead to analytical solution.

A very useful criterion which penalizes long-duration transients is known as the integral of time-weighted by the absolute value of error (ITAE). It is given by

$$J_3=\int_0^\infty t|e(t)|\,dt. \tag{3}$$

This performance index is much more selective than the IAE or the ISE: the minimum value of its integral is much more definable as the system parameters are varied. For a second-order system, this performance index has a minimum for a damping ratio of 0.707.

Other figures of merit which have been proposed are the integral of time-weighted by the squared error (ITSE) and the integral of squared time-weighted by the squared error (ISTSE). These performance indices are defined, respectively, as

$$\text{ITSE}: J_4=\int_0^\infty te^2(t)\,dt, \tag{4}$$

$$\text{ISTSE}: J_5=\int_0^\infty t^2e^2(t)\,dt. \tag{5}$$

For a second-order system, ITSE performance index has a minimum for a damping ratio of 0.594 and ISTSE performance index has a minimum for a damping ratio of 0.653. Data available on the ISTSE criterion indicates that it does result in good responses for systems containing one integration in the open-loop transfer function 2).

In the determination of an optimal constant feedback gain for linear multivariable systems, one can use the following generalized time-weighted quadratic performance index

$$J=\int_0^\infty t^N e^2(t)\,dt. \tag{6}$$

The ITSE and ISTSE performance indices were discussed for single-input single-output systems in 1) and 2). Linear regulators with respect to this subject were considered in 3)~6). A similar problems for dynamic compensators were considered in 7) and 8).

It is well-known that an integral control action is effective in making the output follow a reference input with no steady state error and a derivative control action may give rise to adequate damping of the closed-loop system. In some cases, there exist unstable systems which can not be stabilized through the PID feedback control. Then measurable variable feedback control can be used to increase stability and performance of the closed-loop system. The PID feedback controller using time-weighted quadratic performance index for sampled-data systems has been considered in 7) and the PIM feedback controller for continuous-time systems has been handled in 8) with minimax problem different from that in this paper.

In this paper, optimal constant PIDM feedback controller minimizing a given time-weighted quadratic performance index is considered for continuous-time systems. Necessary conditions for an optimality are derived and the presented result is an unified one which is applicable to several types of controller by removing unnecessary feedbacks.

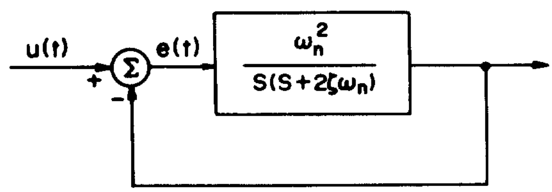

Fig. 1. A second-order system

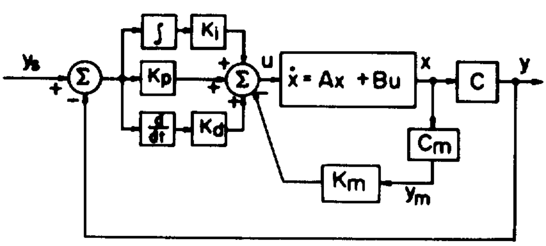

Fig. 2. Block diagram of control system.

2. Problem Fomulation

Consider a linear time-invariant system described by

$$\dot{x}(t)=Ax(t)+Bu(t),\ \ x(0)=0 \tag{7-a}$$
$$y(t)=Cx(t) \tag{7-b}$$
$$y_m(t)=C_m x(t) \tag{7-c}$$

where x is the n-dimensional state vector, u is the r-dimensional control vector, y is the r-dimensional measurable output vector to be controlled, y_m is m-dimensional measurable vector without y, and B, C and C_m have full rank, respectively. It is assumed that the above system is asymptotically stabilizable by control considered below, and that

$$\text{rank of matrix}\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}=n+r. \tag{8}$$

When a step reference input y_s are added to the system, the output y is desired to follow y_s with no steday state error. For this purpose, we consider the following PIDM feedback control law;

$$u=-K_i z-K_p(y-y_s)-K_m y_m-K_d(\dot{y}-\dot{y}_s) \tag{9}$$

where

$$\dot{z}=y-y_s,\ z(0)=0,$$

and K_i K_p K_m and K_d are matrices to be determined. The performance index is given by

$$J=\int_0^\infty [t^N(y-y_s)'Q(y-y_s)+(u-u_s)'R(u-u_s)]dt \tag{10}$$

where Q and R are symmetric positive definite matrix and semidefinite matrix, respectively, and u_s is the steady state control given by the state condition.

The problem is to find out the feedback gain matrices in (9) which minimize the performance index (10).

3. Optimal PIDM Controller

In steady state, state values are uniquely obtainable under the condition of (8) as follows;

$$\begin{bmatrix} x_s \\ u_s \end{bmatrix}=\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ y_s \end{bmatrix}. \tag{11}$$

From (9), u_s also satisfies the following equation;

$$u_s=-K_i z_s-K_m y_{ms}. \tag{12}$$

Using the variables

$$\bar{x}=x-x_s,\ \ \bar{u}=u-u_s,\ \ \bar{y}=y-y_s,\ \ \bar{y}_m=y_m-y_{ms},\ \ \bar{z}=z-z_s, \tag{13}$$

one can obtain the following equations from (7)

$$\dot{\bar{x}}=A\bar{x}+B\bar{u},\ \ \bar{x}(0)=-x_s \tag{14}$$
$$\bar{y}=C\bar{x} \tag{15}$$
$$\bar{y}_m=C_m\bar{x} \tag{16}$$
$$\dot{\bar{z}}=\bar{y}=C\bar{x},\ \ \bar{z}(0)=-z_s \tag{17}$$
$$\bar{u}=-K_i\bar{z}-K_p\bar{y}-K_m\bar{y}_m-K_d\dot{\bar{y}}. \tag{18}$$

The performance index becomes

$$J=\int_0^\infty (t^N\bar{y}'Q\bar{y}+\bar{u}'R\bar{u})\,dt. \tag{19}$$

Now, we define the new state vector and the control vector as

$$w=\begin{bmatrix} \bar{z} \\ \bar{x} \end{bmatrix},\ \ v=\bar{u}. \tag{20}$$

Then, the augmented system is described by

$$\dot{w}(t)=\bar{A}w(t)+\bar{B}v(t), \tag{21}$$

where

$$\bar{A}=\begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix},\ \bar{B}=\begin{bmatrix} 0 \\ B \end{bmatrix},\ w(0)=\begin{bmatrix} -z_s \\ -x_s \end{bmatrix}. \tag{22}$$

The control law (18) gives

$$v=-(I_r+K\bar{D})^{-1}K\bar{C}w = -\bar{K}\bar{C}w \tag{23}$$

where

$$K=[K_i\ K_p\ K_m\ K_d],\ \ \bar{K}=(I_r+K\bar{D})^{-1}K,$$
$$\bar{C}=\begin{bmatrix} I_r & 0 \\ 0 & C \\ 0 & C_m \\ 0 & CA \end{bmatrix},\ \ \bar{D}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ CB \end{bmatrix} \tag{24}$$

Substituting (23) into (21) leads to

$$\dot{w}(t)=Fw(t) \tag{25}$$

where $F=A-\bar{B}\bar{K}\bar{C}$. Then, the performance index becomes

$$J=\int_0^{\infty}(t^N w'\bar{Q}w+w'\bar{C}'\bar{K}'R\bar{K}\bar{C}w)\,dt. \tag{26}$$

where $Q=\mathrm{diag}(O_r,\ C'QC)$.

Theorem 1 : In order that the feedback gain matrix K be optimal with respect to the performance index (10), it is necessary that

$$\begin{aligned} dJ/dK=&2(I+\bar{D}'K')^{-1}\Big[-\bar{B}'\sum_{i=1}^{N+1}(P_iL_i)+R\bar{K}\bar{C}L_{N+1}\Big]\\ &\cdot\bar{C}'(I-\bar{K}'\bar{D}')\\ &-2(K_i')^{-1}(I_r,0_n)P_{N+1}\begin{bmatrix} z_sz_s',\ 0_r,\ z_sx_s'C_m',\ 0_r\\ x_sz_s',\ 0_r,\ x_sx_s'C_m',\ 0_r\end{bmatrix}\\ =&\ 0 \end{aligned} \tag{27}$$

where P_i and L_i satisfy the following equations:

$$F'P_1+P_1F+N!\,\bar{Q}=0 \tag{28-a}$$
$$F'P_{i+1}+P_{i+1}F+P_i=0\ (i=1,\cdots,N-1) \tag{28-b}$$
$$F'P_{N+1}+P_{N+1}F+P_N+\bar{C}'\bar{K}'R\bar{K}\bar{C}=0 \tag{28-c}$$
$$FL_i+L_iF'+L_{i+1}=0\ \ (i=1,\cdots,N) \tag{29-a}$$
$$FL_{N+1}+L_{N+1}F'+w(0)w(0)'=0 \tag{29-b}$$

Then, the final cost becomes

$$J=w(0)'P_{N+1}w(0). \tag{30}$$

Proof of Theorem 1 : Define a Lyapunov function of the following type

$$V(t)=\sum_{i=1}^{N+1}\frac{t^{N-i+1}}{(N-i+1)!}\,w'(t)P_iw(t) \tag{31}$$

where N is the order of the time-weighting factor. Derivative of (31) with respect to time yields

$$\begin{aligned}\dot{V}(t)=&\sum_{i=1}^{N}\frac{t^{N-i}}{(N-i)!}\,w'(t)P_iw(t)\\ &+\sum_{i=1}^{N+1}\frac{t^{N-i+1}}{(N-i+1)!}\,w'(t)[F'P_i+P_iF]w(t).\end{aligned} \tag{32}$$

If rearranged, (32) becomes

$$\begin{aligned}\dot{V}(t)=&\sum_{i=2}^{N+1}\frac{t^{N-i+1}}{(N-i+1)!}\,w'(t)[F'P_i+P_iF+P_{i-1}]\\ &\cdot w(t)+(t/N!)\,w'(t)[F'P_1+P_1F]w(t).\end{aligned} \tag{33}$$

Using (28), one can obtain

$$\dot{V}(t)=-t^Nw'(t)\bar{Q}w(t)-w'(t)\bar{C}'\bar{K}'R\bar{K}\bar{C}w(t). \tag{34}$$

Since F is asymptotically stable, the performance index (10) becomes

$$\begin{aligned}J=&\int_0^{\infty}-\dot{V}(t)\,dt\\ =&V(0)-V(\infty)\\ =&w'(0)P_{N+1}w(0)\\ =&\mathrm{tr}[P_{N+1}w(0)w'(0)].\end{aligned} \tag{35}$$

To determine the optimal feedback gain matrix K which minimizes the performance index (35) subject to the constraints in (28), the Hamiltonian for this problem is defined as follows:

$$\begin{aligned}H(P_i,L_i,K)=&\mathrm{tr}[P_{N+1}w(0)w(0)']+\mathrm{tr}[L_i(F'P_iF+N!\,\bar{Q})]\\ &+\sum_{i=2}^{N-1}\mathrm{tr}[L_{i+1}(F'P_{i+1}+P_{i+1}F+P_i)]\\ &+\mathrm{tr}[L_{N+1}(F'P_{N+1}+P_{N+1}F+P_N+\bar{C}'\bar{K}'\\ &\cdot R\bar{K}\bar{C})]\end{aligned} \tag{36}$$

where L_i, $i=1,\ldots,N+1$, is the symmetric Lagrange multiplier matrix. The necessary conditions for solution are derived by taking the partial dervatives of (36) with respect to P_i, L_i and K, and equating them to zero. The partial derivatives of (36) with respect to P_i, and L_i yield (29) and (28), respectively. The partial derivative of (36) with respect to K also gives rise to (27). Therefore, the proof is completed.

The result in Theorem 1 contains the solutions to several types of control. However, for a special type of control, it has superflous terms and is tedious in form. In the following, necessary conditions for an optimality about various useful control types are considered.

(PI Control)

In this case, $\bar{D}=0$ and $\bar{K}=K$. In order that the PI controller be optimal with respect to the performance index (10), it is necessary that

$$\begin{aligned}dJ/dK=&2\Big[-\bar{B}'\sum_{i=1}^{N+1}(P_iL_i)+RK\bar{C}L_{N+1}\Big]\bar{C}'-2(K_i')^{-1}\\ &\cdot(I_r,\ 0_n)P_{N+1}\begin{bmatrix} z_sz_s', & 0_r\\ x_sz_s', & 0_r\end{bmatrix}=0\end{aligned}$$

where P_i, and L_i satisfy (28) and (29) with the following parameters:

$$K=[K_i,\ \ K_p],\ \ \bar{C}=\begin{bmatrix} I_r & 0\\ 0 & C\end{bmatrix}.$$

(PID Control)

In order that the PID controller be optimal with respect to the performance index (10), it is necessary that

$$dJ/dK = 2(I+\bar{D}'K')^{-1}\left[-\bar{B}'\sum_{i=1}^{N+1}(P_iL_i)+RK\bar{C}L_{N+1}\right] \cdot \bar{C}'(I-\bar{K}'\bar{D}') - 2(K_i')^{-1}(I_r, 0_n)P_{N+1} \cdot \begin{bmatrix} z_s z_s', & 0_r, & 0_r \\ x_s z_s', & 0_r, & 0_r \end{bmatrix} = 0$$

where P_i and L_i satisfy (28) and (29) with the following parameters;

$$\bar{C} = \begin{bmatrix} I_r & 0 \\ 0 & C \\ 0 & CA \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 0 \\ 0 \\ CB \end{bmatrix}$$

$$K = [K_i, \; K_p, \; K_d], \quad \bar{K} = (I_r + K_d CB)^{-1} K.$$

(PIM Control)

In order that the PIM controller be optimal with respect to the performance index (10), it is necessary that

$$dJ/dK = 2\left[-\bar{B}'\sum_{i=1}^{N+1}(P_iL_i)+RK\bar{C}L_{N+1}\right]C' - 2(K_i')^{-1} \cdot (I_r, 0_n)P_{N+1}\begin{bmatrix} z_s z_s', & 0_r, & z_s x_s' C_m' \\ x_s z_s', & 0_r, & x_s x_s' C_m' \end{bmatrix} = 0$$

where P_i and L_i satisfy (28) and (29) with th following parameters;

$$\bar{K} = K = [K_i, K_p, K_m], \quad \bar{C} = \begin{bmatrix} I_r & 0 \\ 0 & C \\ 0 & C_m \end{bmatrix}.$$

A Computational Algorithm

The gradient of J with respect to the unknown elements in K given by (27) is set equal to zero, to get the necessary conditions for a minimum J. The necessary conditions of an optimality cannot directly give the feedback gain K, as they involve P_i, and L_i expressed in terms of K as in (28) and (29), respectively. Moreover, the resulting equations are nonlinear. However, as explicit expressions are available for J and dJ / dK, a gradient technique such as the Davidon-Fletcher-Powell algorithm 9) may be employed to determine K. The computaional algorithm is given below. Let

$$z = \mathrm{col}(K).$$

Step 1 : Find $\bar{K}_o$ so that $(\bar{A}-\bar{B}\bar{K}_o\bar{C})$ is stable. Set i= j=0.

Step 2 : Find $J(K_i)$.

Step 2 : Compute $g_i = \mathrm{col}(dJ/dK)(K_i)$. If $\|g_i\|$ is sufficiently small, stop; otherwise go on.

Step 4 : If $j \neq 0$, define

$$H_i = H_{i-1} + \frac{(z_i - z_{i-1})(z_i - z_{i-1})'}{(z_i - z_{i-1})'(g_i - g_{i-1})} - \frac{H_{i-1}(g_i - g_{i-1})(g_i - g_{i-1})'H_{i-1}}{(g_i - g_{i-1})'H_{i-1}(g_i - g_{i-1})}$$

or else set $H_i = I$; determine

$$s_i = -H_i g_i.$$

Step 5 : Perform a one-dimensional minimization

$$J(z_i + \alpha_i s_i) = \min_{\alpha \geq 0} J(z_i + \alpha s_i).$$

Let $z_{i+1} = z_i + \alpha_i s_i$ and i=i+1. If j= 2dim[z], set j= 0, or else j=j+1.

Return to step 3.

4. A Numerical Example

The buck type switching regulator in Fig. 3 is considered. To obtain the state differential equation for this analysis, the state space averaging concept has been adopted 10)~11). With the choice of the state vector,

$$x(t) = [V_m(t) \;\; V_o(t) \;\; i_{L1}(t) \;\; i_{L2}(t)]'$$

one can obtain the following system matrices;

$$A = \begin{bmatrix} -R_3(1/L_1+1/L_2) & R_3/L_2 & 1/C_1 - R_1R_3/L_1 & R_2R_3/L_2 - 1/C_1 \\ R_LR_4/[L_2(R_L+R_4)] & \frac{-R_L(R_4/L_2+1/R_LC_2)}{(R_L+R_4)} & 0 & \frac{R_L(1/C_2 - R_2R_4/L_2)}{(R_L+R_4)} \\ -1/L_1 & 0 & -R_1/L_1 & 0 \\ 1/L_2 & -1/L_2 & 0 & -R_2/L_2 \end{bmatrix}$$

$$B = [R_3/L_1 \;\; 0 \;\; 1/L_1 \;\; 0]'$$

$$C = [0 \;\; 1 \;\; 0 \;\; 0]$$

$$C_m = [1 \;\; 0 \;\; 0 \;\; 0]$$

where y(t) and $Y_m(t)$ denote output voltage and measurable output voltage, without respectively.

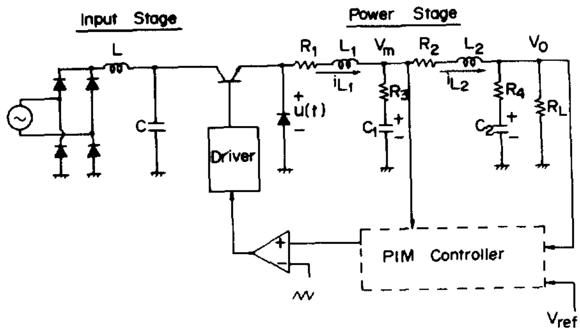

Fig. 3. Circuit diagram of the buck type switching regulator.

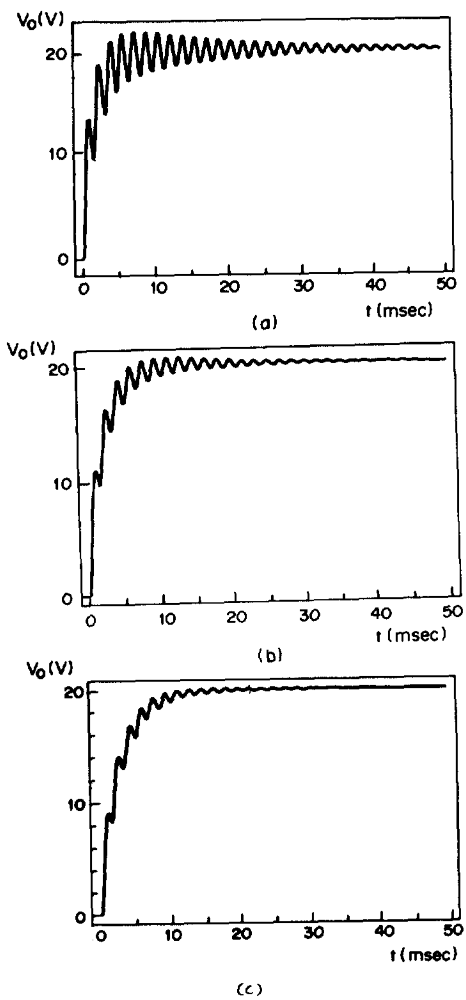

Fig. 4. Tranisent responses of the closed-loop system (a) N= 0, (b) N= 2. (c) N= 5.

Table. 1. Optimal constant gains and settling times (t_r=0.005 sec., R=0.2).

time weighting index (N)	optimal gains K_i I_p I_m	settling time (msec)
0	7211 1.91 11.7	22
2	8931 1.00 18.3	10
5	6827 1.12 18.8	9.3

We consider the following PIM controller;

$$u(t) = -K_i z(t) - K_p[y(t) - y_s] - K_m y_m(t)$$
$$\dot{z}(t) = y(t) - y_s, \quad z(0) = 0$$

and adopt the following performance index

$$J = \int_0^\infty [(t/t_r)^N \bar{y}^2 + R\bar{u}^2]\, dt$$

where N is positive integer. Then, the weighting matrix Q becomes as

$$\bar{Q} = \text{diag}(0, \ 0, \ 1/t_r^N, \ 0, \ 0)$$

were t_r in the performance index is used to give relatively larger weighting on the sustained output error after expected rising time. System parameters are as follows;

$L_1 = L_2 = 0.3$mH, $R_1 = R_2 = 0.015$ohm
$C_1 = C_2 = 330\mu$F, $R_3 = R_4 = 0.1$ ohm
$R_L = 10$ ohm, $V_{ref} = y_s = 20$V.

The optimal PIM feedback gains from the proposed algorithm are derived as shown in Table 1 and the transient responses of the output voltage are simulated as shown in Fig. 4. As can be seen in Table 1, 5% settling time of output response using the time-weighted quadratic performance index (N= 2, 5) is shorter than that of using the conventional quadratic performance index (N=0) due to the oscillatory response.

5. Conclusion

In this paper, necessary conditions to be satisfied by an optimal constant PIDM controllers using a class of time-weighted quadratic performance indices have been derived for linear time-invariant multivarible systems. This design method of the regulators using the time-weighted quadratic performance index provides better performance characteristics in the time domain compared to the one using conventional quadratic performance index. This design technique has been utilized to improve the transient performance for the buck-type switching regulator. Computer simulations have also been given to show the usefulness of this technique. An algorithm for computing the optimal constant feedback gain has been presented.

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