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RECENT DEVELOPMENTS IN DIFFERENTIAL GEOMETRY AND MATHEMATICAL PHYSICS

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I want to focus on developments in the areas of general relativity and gauge theory. The topics to be considered are the singularity theorems of Hawking and Penrose, the positivity of mass, instantons on the four-dimensional sphere, and the string picture of quantum gravity. I should mention that I will not have time to discuss either classical mechanics or symplectic structures. This is especially unfortunate, because one of the roots of differential geometry is planted firmly in mechanics, Cf. [GS].

The French geometer Elie Cartan first formulated his invariant approach to geometry in a series of papers on affine connections and general relativity, Cf. [C]. Cartan was trying to recast the Newtonian theory of gravity in the same framework as Einstein's theory. From the historical perspective it is significant that Cartan found relativity a convenient framework for his ideas. At about the same time Hermann Weyl in troduced the idea of gauge theory into geometry for purposes much different than those for which it would ultimately prove successful, Cf. [W]. Weyl wanted to unify gravity with electromagnetism and though that a conformal structure would fulfill the task but Einstein rebutted this approach.

1. General relativity

Let M denote a smooth four-manifold with Lorentzian structure $g_{\alpha\beta}$, whose index is one. M will be referred to as-spacetime. Thus in normal co-ordinates at a point P the form of $g_{\alpha\beta}$ is diag$(-1, 1, 1, 1)$. (Lower case Greek letters used as indices will always have the range $0, 1, 2, 3$.) This proves that special relativity is still valid even if gravitational effects are ignored. Although the Lorentzian structure does

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not have the same local topological implications that a Riemannian (positive definite $g_{\alpha\beta}$) structure has, it still yields arclength for smooth curves and a separation of regular curves into timelike, spacelike, and null categories, depending on the nature of the tangent vector field T to the curve. A timelike curve, in which $g(T, T)$ is negative, corresponds to a material particle traveling at an observed velocity less than the speed of light. So far particles tracing out spacelike curves, in which $g(T, T)$ is positive, have not been observed.

To have a coherent theory of gravity a replacement for Newton's universal law of gravitation must be found. This is somewhat complicated in Einstein's theory because the $g_{\alpha\beta}$ are in fact the gravitational field. After a great deal of hard work Einstein finally phrased his now famous system of equations in terms of curvature and external physical fields present.

To understand the notion of curvature, start with the replacement for straight lines in a Lorentzian manifold, its geodesics. These curves have the property that their tangent vectors are parallel translated into themselves along the curve. Curvature can now be interpreted as the property of the spacetime that tells whether or not a pencil of geodesics will focus or diverge. In particular, the Ricci curvature tells whether or not geodesics starting orthogonal to some three-dimensinal hypersurface will focus or diverge. Ricci curvature is a tensor of the same rank as the Lorentzian structure, so denote Ricci by $R_{\alpha\beta}$. The distribution of matter over the space time is given by a tensor $T_{\alpha\beta}$, which is symmetric in the indices α, β. $T_{\alpha\beta}$ will change, depending on the type of matter present. Now the Einstein equation can be stated as:

$$R_{\alpha\beta} - (1/2) R\, g_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{E}$$

in which R is the trace of $R_{\alpha}{}^{\beta}$. This equation was derived by Hilbert from a variational principle, at the same time as Einstein, but Hilbert was careful to point out his debt to Einstein for intoducing him to this problem.

The difficulty with equation (E) is that it is a quasilinear system of hyperbolic equations in $g_{\alpha\beta}$, and hence there are general procedures for fining solutions. But there is nothing, at present, built into relativity to limit solutions to (E). Each solution must be examined and accepted or rejected on rejected on the basis of physical relevance. Many explicit solutions of (E) are known, some having been found almost immediately after Einstein's formulation of the general theory. Two solutions are of particular interest–the Schwarzschild solution and

The Friedmann–Robertson–Walker solutions. The Schwarzschild solution becomes the model for a local gravitational system that will eventually form a black hole, while the FRW solutions are thought of as cosmological solutions or solutions for the entire history of the universe:

(S) $$ds^2 = -(1+c/r)dt^2 + (1+c/r)^{-1}dr^2 + r^2 d\sigma^2$$

(FRW) $$ds^2 = dt^2 + f(t)^2 d\Sigma^2$$

in which c is a constant, $d\sigma^2$ is the metric of the two–sphere, and $d\Sigma^2$ is the metric of a three–manifold of constant curvature. The function $f(t)$ in (FRW) is calculated from (E) using an energy–momentum tensor $T_{\alpha\beta}$ for a dust cloud of galaxies. The energy momentum tensor for (S) is assumed zero so the solution fits the exterior of a collapsing spherically symmetric star. The functions $f(t)$ in (FRW) give the familiar picture of a universe, expanding from a "big bang". In the case of positive spatial curvature, the expansion continues until the hypersurface has maximal volume, at which point contraction begins, ending ultimately in a "big crunch". In the case of negative spatial curvature, the universe tapers–off at infinity and so gives the appearance of freezing to death. For some time in general relativity, it was felt that the extremes in these solutions and the ultimate dire consequences, could be avoided in a generic unsymmetric model. These hopes were dashed by a spectacular series of theorems by Hawking and Penrose that launched relativity into a new era. Typical of these facts is:

Theorem: If the value of the Ricci curvature applied to any timelike vector is non–negative and if there is a compact spacelike hypersurface with positive mean curvature, then the spacetime contains a timelike geodesic whose affine parameter cannot be extended to the entire real line.

In other words a particle in free fall along that path could veer off to oblivion or meet an untimely end. It remains a challenge to figure out what exactly goes wrong in this situation.

The positive mass problem is somewhat more subtle. For a metric which is like Schwarzschid, that is, asymptotically flat, there is an associated mass–the so–called ADM–mass, named for Arnowit, Deser and Misner. In the case of the Schwarzschid solution the constant of integration c is the ADM–mass. The general problem is to determine the positivity of the ADM–mass on the basis of the positivity of Ricci on time like vectors. This was accomplished in a very beautiful way by

Schoen and Yau. A short time later Edward Witten gave a very elegant proof based on spinors.

2. Gauge theory

Here the situation is a little different because the fields are placed on top of the spacetime, not woven into its fabric as in general relativity. Maxwell's theory of electromagnetism serves as the model from which Yang and Mills originally based their theory, which was formulated to explain isotopic spin. The vector potential is denoted by A and the electromagnetic field is represented by $F=dA$, in which d is the exterior derivative and as a result F is a differential 2-form. In the case of a source-free field, Maxwell's equation reads:

(M) $$\delta F=0$$

in which δ is the divergence operator. After some thought it was recognized that Maxwell's theory could be described mathematically as a circle bundle, in which a connection had been chosen, over spacetime. The vector potential is the connection form, explaining the peculiar way that A transforms under change of fiber co-ordinates, and the field is given by the curvature form.

In generalizing these ideas to bundles with other groups, assume that the spacetime has been replaced by a Riemannian four-manifold M. This situation is also of interest to physicists as it means, in quantum theory, that ordinary time, t, has been replaced by imaginary time, it. This has the benefit of making the Hodge star operator easier to deal with than in the Lorentzian index. Recall that the star operator takes a k-plane in the tangent space into the orthogonal $(4-k)$-plane, suitably oriented. On bivectors, the square of the star operator is the identity (the manifold has dimension four!). With this background information out of the way, consider a vector bundle E over M, in which the group acting on the fibers is $SU(2)$. Now provide E with a connection A, as in Maxwell's theory, but notice that the curvature form will reflect the non-abelian nature of $SU(2)$ and as a result $F=dA+A\wedge A$. The Yang-Mills equation for the connection A is then:

(YM) $$D^{*}F=0$$

in which D^{*} represents the formal adjoint of the covariant derivative corresponding to A. A special class of solutions can be obtained by observing that $DF=0$ for any conection. This is Bianchi's identity. If in addition F satisfies the non-linear equation $*F=F(-F)$ then F is automatically a solution to (YM). Such solutions are called instantons (anti-instantons). Assuming gauge equivalence of A to a flat connection at infinity, which is a boundary condition at infinity, the connection can be extended to the four-dimensional sphere.

The basic instanton on $S^4=P_1(H)$, the quaternionic projective line, can be constructed by considering the tautological line bundle, E_t. To construct E_t take the subbundle of the trivial bundle on $P_1(H)$ described by $\{([q^0, q^1],\ (a^0, a^1)):\ (a^0, a^1)\in \mathrm{span}(q^0, q^1)\}$. Consider the connection on E_t by pulling back the Kaehler form from $P_1(H)$. In inhomogeneous co-ordinates the curvature form can be represented by:

$$F=dx\wedge d\bar{x}/(1+|x|^2)^2$$

in which x represents a quaternion. It is not difficult to check that the second Chern class of E_t is minus one. All of the solutions, with Chern class minus one, can be obtained from the solution above by quaternionic fractional linear transformations of the form:

$$x \longrightarrow \mu(x-b)$$

in which μ is real and b is an arbitrary quaternion.

Physically, instantons are thought to describe an aspect of the strong force of interaction in particle physics, but it came as a great surprise that instantons have differentiable implications for ordinary 4-dimensional Euclidean space. S. Donaldson, in 1982, proved, using the work of Taubes on instantous, that if the intersection form of a simply connected 4-manifold is definite then it is diagonalizable over the integers. Michael Freedman then put this to use in showing that R^4 had more than one smooth structure.

3. Quantum gravity

In this final section, a summary will be given of current work of Candalas, Horowitz, Strominger, and Witten on an approach to quantum gravity. In general terms a physical theory is not considered complete until it can be "quantizd". In the case of general relativity this process faces insurmountable difficulties, if the standard approach used in electrodynamics is applied. For this reason physicists have sought a resolution of the problem by introducing new techniques. To avoid a long discussion of basic underlying physical principles, only the mathematically relevant part of the treatment will be covered. The situation comes down to considering a manifold of the form $M_4 \times K$ in which M_4 is maximally symmetric and K is some compact 6-dimensional manifold. These extra dimensions are thought to be small in the sense that radius of K should be of the order of magnitude of the Planck length. The physical assumption of the existence of what is called an unbroken supersymmetry implies that M_4 must be flat and that K have holonomy in $SU(3)$. It follows then that K is a complex Kaehler manifold. S. T. Yau has recently shown the existence of spaces with $SU(3)$ holonomy, a candidate for which is the Kummer surface:

$$z_1^5 + \cdots + z_5^5 = 0$$

in $P_4(C)$. Thus Yau's theorem tells us that the Kummer surface has a Kaehler structure whose Ricci curvature vanishes. In addition, the unbroken supersymmetry provides the existence of an Einstein-Hermitian vector bundle over K. Finally, an alternate viewpoint, based on the notion of a string, leads to similar conclusions. Normally, a particle propogates along a timelike curve, which is a geodesic when there is no force acting. What happens if in place of a point, a one dimensional "string" is allowed to propogate? If the string were a spacelike curve to begin with, it should propogate in such a way that it traces out a timelike surface. To replace the geodesic condition for a curve, the mean curvature vector of the surface should vanish. Such a timelike surface would then be a maximal surface from the viewpoint of the area functional. The details of how the string idea is applied to the situation at hand can best be found in the work of Candalas, Horowitz, Strominger and Witten.

It is a source of wonder that so many different ideas from differential geometry should come to bear on the basic physics of our universe.

References

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[GS] V. Guillemin and S. Sternberg, *Geometric Asymptotics*, Providence, 1977.

[W] H. Weyl, *Gravitation und Elektrizitaet*, Sitz. der Koenig. Preus. Akad.

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Further Reading

The basic ideas as well as the singularity theorems of general relativity can best be found in Robert Wald's book on General Relativity, Chicago, 1984 and Barrett O'Neill's book on Semi-Riemannian Geometry, New York, 1983. A more thorough discussion of the positive mass theoremis elucidated by G. Horowitz in an article in Lecture Notes in Physics, V. **202**, Berlin, 1984.

An excellent introduction to gauge theory is given in the Fermi lectures of Michael Atiyah, Geometry of Yang-Mills Fields, Pisa, 1979. A discussion of the exotic differentiable structures on R^4 can be found in Michael Freedman's ar ticle in the Notices of the AMS, V. **31**, (1984), 3-6.

The summary given in part 3 comes from the paper: Vacuum configurations for superstrings, by P. Candalas, G. Horowitz, A. Strominger and E. Witten, in Nuclear Physics B258 (1985(, 46-74. Background material for the last section can be best gotten from S. Kobayashi's forthcoming book: Differential Geometry of HolomorphicVector Bundles.

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