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ON A PROBLEM OF HALMOS ABOUT INVERTIBLE OPERATORS

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1. Introduction

If A is a bounded linear operator on a Hilbert space, and if the operator norm $\|I-A\|<1$, then A is invertible, see Halmos [2, p. 52]. That assertion, not only for operators, on Hilbert spaces, but for arbitrary elements of Banach algebras, is usually proved by consideration of the infinite series $S=\sum_{n=0}^{\infty}(I-A)^n$, that is,

$$AS=(I-(I-A))S=\sum_{n=0}^{\infty}(I-A)^n-\sum_{n=1}^{\infty}(I-A)^n=I,$$

see Dunford and Schwartz [1, p. 585]. In this operation, the completeness is sufficient. Recently, in [3], Halmos asked the question as to whether the assertion is true without completeness? In other words, is there a bounded linear operator A on an inner product space, such that $\|I-A\|<1$ but A is not invertible? The answer turns out to be affirmative as follows:

THEOREM 1. *There is a bounded linear operator A on an incomplete inner product space such that $\|I-A\|<1$ but A is not invertible.*

2. Proof of Theorem 1

We let V be the real vector space generated by the sequence x^n, $n=0, 1, \cdots$, where $0\leq x\leq 1$, and let the inner product be defined by

$$(\mathrm{f}, \mathrm{g})=\int_0^1 f(x)g(x)dx, \quad \text{for any } f, g\in V.$$

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Since the space V consists of only finite combinations of x^n, the function $1/(1+x)=1-x+x^2\cdots\notin V$ so V is an incomplete inner product space. We set $h(x)=(1+x)/2$ and define the operator $A_h(f)=hf$ for each $f\in V$. Then clearly A_h is a bounded linear operator. Since $(I-A_h)(f)=(1-h)f=\frac{1-x}{2}f$, it follows that

$$\|I-A_h\|=\sup_{\|f\|\le 1}\left\{\int_0^1\left(\frac{1-x}{2}f(x)\right)^2dx\right\}^{\frac{1}{2}}\le\frac{1}{2}.$$

Now, the element $1\in V$, but $1/h\notin V$, and hence the operator A_h is not invertible in V. This shows that the condition $\|I-A\|<1$ is not sufficient for an operator to be invertible in an incomplete space and the proof is complete.

3. Necessity and sufficiency of invertibility

Naturally, we may ask the necessary and sufficient condition for an operator A with $\|I-A\|<1$ to be invertible in a space (complete or not). This question will be answered by the following

THEOREM 2. *Let A be a bounded linear operator in a norm space V (complete or not) and let $\|I-A\|<1$. Then A is invertible if and only if $A(V)=V$.*

Proof. If A is invertible, we denote its inverse by A^{-1}. Suppose on the contrary that there is an element $x\in V$ but $x\notin A(V)$. Then $AA^{-1}(x)\neq x$, a contradiction. This shows that $A(V)=V$.

Conversely, if $A(V)=V$, we shall prove that A is one-to-one. Again, suppose $A(x)=A(y)$ for some $x\neq y$ in V. Let $u=(x-y)/\|x-y\|$, then $\|u\|=1$ and $(I-A)(u)=u$, so that $\|I-A\|\ge\|u\|=1$, a contradiction. Hence A is one-to-one and the inverse A^{-1} exists.

It remains to prove that A^{-1} is bounded. We shall give two proofs due to B. Shekhtman and R. Ryan respectively.

First proof. Let $\|I-A\|=\rho<1$, then $\|x-Ax\|\le\rho\|x\|$ for each $x\in V$. Since $AA^{-1}x=x$ we have

$$\|A^{-1}x\|-\|x\|\le\|A^{-1}x-AA^{-1}x\|\le\rho\|A^{-1}x\|,$$

so that

$$\|A^{-1}x\|\le(1-\rho)^{-1}\|x\| \text{ and } \|A^{-1}\|\le(1-\rho)^{-1}.$$

Second proof. Let $\bar{V}$ be the completion of V and let $\bar{A}$ be the extension of A in $L(\bar{V})$. Then we have $\|I-\bar{A}\|<1$ so that $\bar{A}^{-1}$ exists and bounded in $L(\bar{V})$. Hence the restriction $A^{-1}=\bar{A}^{-1}|_V$ is bounded.

As a consequence, we obtain the following criterion of invertible operators.

COROLLARY. *Let A be a bounded linear operator in a complete space V and let $\|I-A\|<1$. Then A is invertible.*

Proof. For each $x\in V$, the completeness of V guarantees the vector $S(x)=y\in V$, where $S=\sum_{n=0}^{\infty}(I-A)^n$. Since $A(y)=x$, so that $A(V)=V$ and the assertion follows from Theorem 2.

References

1. N. Dunford and J. T. Schwartz, *Linear operators, Part I: General theory,* Interscience, New York 1958.
2. P. R. Halmos, *Introduction to Hilbert space and the theory of spectral multiplicity*, Chelsea, New York 2e. 1957.
3. P. R. Halmos, *Advanced problems,* Amer. Math. Monthly, **90** (1983), 289.

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