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ON INTEGRAL MEANS OF DERIVATIVES OF UNIVALENT FUNCTIONS

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Let S denote the class of univalent functions normalized so that $f(0)=f'(0)-1=0$ in $|z|<1$. Let S_α^*, $-\pi/2<\alpha<\pi/2$, denote the subclass of S that satisfies Re $e^{i\alpha}zf'(z)/f(z)\geq 0$ in $|z|<1$; then f is called α-spiral-like and the case $\alpha=0$ is the class of normalized starlike functions [6, p. 52]. Let T denote the class of functions f normalized as above and satisfying Im z[Im $f(z)$]≥ 0 in $|z|<1$; then f is called typically real and T contains those functions of S whose coefficients are real [6, p. 55].

Also, in view of [6, p. 231], let $B(\lambda)$ be the class of function normalized as above and map $|z|<1$ onto the complement of an arc with radial angle $\lambda(0<\lambda<\pi/2)$. The radial angle is meant to be the angle between the tangent and radial vectors to the arc. This note includes a sharp version for Corollary 1 of [2] when $f\in S_\alpha^*$ as well as a logarithmic coefficient estimate.

1. Integral means

THEOREM 1. *Let $f\in S_\alpha^*$ and $k(z)=z(1-z)^{-2}$; then for $z=re^{i\theta}$ $(0<r<1)$ and $-\infty<p<\infty$ we have*

$$\int_0^{2\pi}|zf'(z)/f(z)|^p d\theta\leq\int_0^{2\pi}|zk'(z)/k(z)|^p d\theta,$$

$$\int_0^{2\pi}|f'(z)|^p d\theta\leq\int_0^{2\pi}|k'(z)|^p d\theta.$$

Proof. The second inequality is due to Leung (see [6, Theorem 7.2]) and the proof of the first inequality is a straightforward application of Baernstein's $*$-function argument (see [2, Theorem 1], [6, Chapter 7]). Note that from the definition of S_α^* above, we may write $e^{i\alpha}zf'(z)/f(z)=\mathrm{F}(z)$ where $\mathrm{F}(z)$ is a function with positive real part (see [7, Vol. I, p. 148]). Thus, as in the proof of Theorem 7.2 of [6],

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we have

$$\left[\pm\log\left|\frac{zf'(z)}{f(z)}\right|\right]^* = (\pm\log|e^{-i\alpha}F(z)|)^* \leq \left[\log\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|\right]^*$$
$$= \left[\pm\log\left|\frac{zk'(z)}{k(z)}\right|\right]^*.$$

Our inequality now follows by setting $(u) = \exp[pu]$ in Lemma 5 of [6].

REMARK 1. Theorem 1 holds for the class T (note that functions in T are not necessarily univalent) by the argument used in [9] and the inclusion properties mentioned in [4, p. 92] and [9, p. 374]. Namely that T is contained in the closed convex hull of $S_0{}^*$.

THEOREM 2. *Let $f \in S$ and $k(z) = z(1-z)^{-2}$; then we have for $p=1, 2, 3, \cdots$ that*

$$\int_0^{2\pi} |f'(z)|^{2p} d\theta \leq \int_0^{2\pi} |k'(z)|^{2p} d\theta.$$

Proof. We see from De Brange's celebrated result [5] that $f'(z) \ll (1+z)(1-z)^{-3}$ where, as in [7, Vol. II, Theorem 5], $\ll$ means that if $\sum c_n z^n \ll \sum D_n z^n$ then $|c_n| \leq D_n$, $n=1, 2, 3, \cdots$. Thus, for a positive integer p we have $[f'(z)]^p \ll [(1+z)/(1-z)^3]^p$. This implies that if $[f'(z)]^p = \sum c_n(p) z^n$ and $[(1+z)/(1-z)^3]^p = \sum D_n z^n$, then $|c_n(p)| \leq D_n(p)$, where $n=1, 2, 3, \cdots$. Using this and Parseval's identity, we deduce that

$$\int_0^{2\pi} |f'(z)|^{2p} d\theta = 2\pi \sum_{n=0}^{\infty} |c_n(p)|^2 r^{2n} \leq 2\pi \sum_{n=0}^{\infty} D_n{}^2(p) r^{2n}$$
$$= \int_0^{2\pi} |k'(z)|^{2p} d\theta$$

as required in Theorem 2.

THEOREM 3. *Let $F(z)$ satisfy Re $F(z) \geq 0$ and $F(0)=1$ in $|z|<1$; then for $k(z) = (1+z)(1-z)^{-1}$ and $-\infty < p < \infty$ we have*

$$\int_0^{2\pi} |F(z)|^p d\theta \leq \int_0^{2\pi} |k(z)|^p d\theta,$$
$$\int_0^{2\pi} |F'(z)|^p d\theta \leq \int_0^{2\pi} |k'(z)|^p d\theta$$

where $z = re^{i\theta}$ and $0<r<1$.

This theorem fills the gap in [7, Vol. II, p. 20], [8, Theorem 4] and [9, p. 373]. We omit the proof of this theorem since it follows exactly the proof of Theorem 3 by applying the $*$-function.

2. Logarithmic coefficients estimate

THEOREM 4. *Let* $f\in B(\lambda)$ *and* $\log f(z)/z=2\sum_{n=1}^{\infty}\gamma_n z^n$. *Then for* $n\geq 1$ *we have*

$$|\gamma_n|\leq A(\lambda)/n,$$

where $A(\lambda)$ *is a constant depending on* λ *only and not necessarily the same whenever it occurs in the rest of this note.*

To prove Theorem 4 we need the following result.

LEMMA. *Let* $f\in B(\lambda)$. *Then for* $z=re^{i\theta}$ $(0<r<1)$ *we have*

$$\left.\begin{array}{l}\displaystyle\int_0^{2\pi}\left|\frac{z^2f''(z)}{f(z)}\right|d\theta\\ \displaystyle\int_0^{2\pi}\left|\frac{zf''(z)}{f'(z)}\right|^2 d\theta\end{array}\right\}\leq\frac{A(\lambda)}{1-r}.$$

Proof. In view of [2, p. 340] we may define $f(z)$ by the equation

$$izf'(z)/f(z)=(e^{i\phi}-z)g(z)/(1-z) \tag{1}$$

where $0<\phi<2\pi$ and $g\in B(\lambda)$ for some $\lambda\in(0,\pi/2)$. Differentiating this, rearranging, taking the modulus bound and integrating with respect to θ $(0\leq\theta\leq 2\pi)$, we deduce, for $z=re^{i\theta}$ $(0<r<1)$, with the help of Schwarz's inequality and Corollary 1 of [2] that

$$\begin{aligned}\int_0^{2\pi}\left|\frac{z^2f''(z)}{f(z)}\right|d\theta &\leq\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2 d\theta\\ &+\left[\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2 d\theta\right]^{\frac{1}{2}}\left\{\left[\int_0^{2\pi}\left|\frac{zg'(z)}{g(z)}\right|^2 d\theta\right]^{\frac{1}{2}}+\left[\int_0^{2\pi}\left|\frac{z}{1-z}\right|^2 d\theta\right]^{\frac{1}{2}}\right.\\ &\left.+\left[\int_0^{2\pi}\left|\frac{1}{e^{i\theta}-z}\right|^2 d\theta\right]^{\frac{1}{2}}\right\}\\ &\leq A(\lambda)/(1-r)\end{aligned}$$

as required. Note that $g(z)\in B(\lambda)$ and hence Corollary 1 of [2] is applicable.

Similarly we deduce the second inequality of the lemma from (1).

Proof of Theorem 4. We see from the hypothesis of this theorem that $[zf'(z)/f(z)]'=2\sum_{n=1}^{\infty}n^2\gamma_n z^{n-1}$. Applying the coefficients formula to this we deduce, using Schwarz's inequality and the lemma, that

기관지를 增設하여 論文 爲主, 外國 Journal과의 交換, 交流 爲主의 "大韓數學會會誌"(Journal of the Korean Mathematical Society, 略하여 Jour, K. M. S.)와 硏究發表會에서 公表된 것을 主로 한 一般 論文, 記事, 討論會記事, 論說, 書評 等을 揭載하는 國內情報交換 爲主의 大韓數學會會報(Bulletin of the Korean Math. Soc. 略하여 Bull, K. M. S.)로 2分하여 各各 年 2回 出刊(全四册)하기로 되어 本學會는 一大飛躍의 첫 발을 내 딛은 것입니다. 여러분이 잘 아시는 靑色表紙의 회지와 黃色表紙의 회보는 이때 비로소 시작된 것입니다. 勿論 本學會는 現在는 또 하나의 기관지 "대한 수학회 논문집"(Communications of the K. M. S.)을 今年 1986年부터 刊行하게 되어 이제는 3種의 어디 내어도 손색없는 기관지를 갖게 되었읍니다만, 이런 意味에서도 1970年는 記念할 만한 해라고 생각됩니다. 더구나 今年에는 40주년을 記念하여 以上의 Journal과 Bulletin 揭載의 全論文의 影印本이 刊行되었으니 참으로 기쁜 일이라 아니할 수 없읍니다.

1970年부터는 對外的으로도 學會의 活動의 門이 활짝 열렸다고 생각됩니다. 여기서는 그 中의 몇 가지 만을 回想키로 하겠읍니다. 1971年 3月에는 B. Volkmann 교수와의 Seminar (Hausdorff measure)가, 4月에는 W. Elli 교수의 초청강연, 7月에는 H. Matsumura 교수, 같은 이달에 在美中의 林德相교수(故)의 초청 세미나(Algebraic geometry)가 있었고, 1972年 3月에 역시 대수기하학에 관하여 H. Hironaka의 초청강연이 있는 等, 其他는 省略하겠읍니다만 매우 活潑하게 交流가 이루어졌읍니다. 이러한 行事 뒤에는 여러모로 財政的인 뒷받침이 要請되는 법인데, 金正洙 會長을 비롯하여 理事會 任員, 特別會員 여러분들의 숨은 功勞는 길이 남을 것입니다.

1974年에 접어들어 學會는 從來의 體制였던 任意團體를 脫皮하여 待望의 社團法人體에로의 改編作業에 들어가서 여러분이 잘 아시는 바와 같이 現在 本學會는 科學技術處를 主務官廳으로 하는 "社團法人大韓數學會"로 되어 있음은 再言할 必要가 없겠읍니다만, 이에 따른 附隨 事業도 亦是, 順調롭게 進行되어 學會基金育成關係, 學會事務室購入關係等 權宅淵, 朴世熙, 趙泰根 여러 歷代 會長의 盡力은 勿論, 많은 特別會員들의 支援, 特히 洪性大會員을 잊지 못하는 터이지만 많은 분들의 덕분으로 着着成果를 올렸으니, 이는 全會員과 더불어 同慶하여 마지 않습니다.

이 機會에 빠뜨릴 수 없다고 생각되는 한 두가지를 들먹이 보겠읍니다. 그 하나는 1979年에 있었던 行事인데, 서울대 數學科와 本學會가 共同 主管으로, AID의 援助를 얻어서 가졌던 "韓·美 數學웍샾"(Korea-U. S. Math. -Workshop '79)이라는 것인데 國外에서는 H. Hironaka(Harvard Univ.), F. Raymond (U. of Michigan), F. Tréves (Rutgers U.), Dock S. Rim (U. of Penn), K. W. Kwun (Michigan St. U.), C. N. Lee (〃), B. Y. Chen (〃) 等을 위시한 10여명이 초청되었었고, 國內의 各 大學에서 約 150餘名의 會員 數學者가 參加한 近 10日間에 걸친 大

Tampa (1983).
8. D. Hallenbeck and T. H. MacGregor, *Integral estimates for the derivatives of functions in families of defined through subordination*, Complex Variables **3**(1984), 135-167.
9. T. H. MacGregor, *Application of extreme-point theory to univalent functions*, Michigan Math. J. **19**(1972), 361-376.

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