

## FIXED POINTS FOR FAMILY OF MAPPINGS\*

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Common fixed point theorems for family of mappings on 2-metric spaces (Gähler 1963/64) have been established among others by Lal-Singh (1978), Rhoades (1979), Singh (1979), Ram (1982) & Cho (1985). Ram (1982) established the following result:

**THEOREM 1.** Let  $\{S_n\}$  be a sequence of mappings from a complete 2-metric space  $X$  to itself. Let  $T$  be a continuous mapping from  $X$  to itself such that  $T$  and  $S_n$  commute and  $S_n(X) \subseteq T(X)$ ,  $n=1, 2, 3, \dots$ . If there exists a positive number  $q < 1$  such that for every pair  $i, j$ ,  $i \neq j$ ,

$$(1.1) \quad d(S_i x, S_j y, a) \leq q \cdot \max \left\{ d(Tx, Ty, a), \right. \\ \left. d(S_i x, Tx, a), d(S_j y, Ty, a), \right. \\ \left. \frac{1}{2} [d(S_i x, Ty, a) + d(S_j y, Tx, a)] \right\}$$

for all  $x, y, a$  in  $X$ , then  $T$  and the sequence  $\{S_n\}$  of mappings have a unique common fixed point. We prove the following:

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\* The intent of this paper is to offer common fixed point theorems for a countable family of mappings on 2-metric spaces.

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THEOREM 2. Let  $\{S_n\}$  be a sequence of mappings from a 2-metric space  $X$  to itself. Let  $T$  be a mapping from  $X$  to itself such that

$$(2.1) \quad S_n(X) \subseteq T(X), \quad n=1, 2, 3, \dots,$$

$$(2.2) \quad T(X) \text{ is a complete subspace of } X.$$

If there exists a positive number  $q < 1$  such that for every pair  $i, j$ ,  $i \neq j$ , the condition (1.1) holds, then  $T$  and  $S_n$  ( $n=1, 2, 3, \dots$ ) have a coincidence point, i.e. there exists a point  $z$  such that

$$Tz = S_n z, \quad n=1, 2, 3, \dots$$

Further, if  $T$  and  $S_n$ ,  $n=1, 2, 3, \dots$ , commute at  $z$  then  $T$  and  $S_n$ ,  $n=1, 2, 3, \dots$ , have a unique common fixed point. Indeed  $Tz$  is the unique common fixed point.

PROOF. Pick  $x_0$  in  $X$ . Construct a sequence  $\{Tx_n\}$  such that  $Tx_n = S_n x_{n-1}$ ,  $n=1, 2, 3, \dots$ . We can do this since  $S_n(X) \subseteq T(X)$ . By (1.1),

$$\begin{aligned} d(Tx_{n+1}, Tx_n, a) &= d(S_{n+1}x_n, S_n x_{n-1}, a) \\ &\leq q \cdot \max \{ d(Tx_n, Tx_{n-1}, a), \\ &\quad d(S_{n+1}x_n, Tx_n, a), \\ &\quad d(S_n x_{n-1}, Tx_{n-1}, a), \\ &\quad \frac{1}{2} [d(S_{n+1}x_n, Tx_{n-1}, a) \\ &\quad + d(S_n x_{n-1}, Tx_n, a)] \} \\ &= q \cdot \max \{ d(Tx_n, Tx_{n-1}, a), \\ &\quad d(Tx_{n+1}, Tx_n, a), \\ &\quad d(Tx_n, Tx_{n-1}, a), \\ &\quad \frac{1}{2} [d(Tx_{n+1}, Tx_{n-1}, a) \\ &\quad + d(Tx_n, Tx_n, a)] \}, \end{aligned}$$

giving

$$d(Tx_{n+1}, Tx_n, a) \leq q \cdot \max \left\{ d(Tx_n, Tx_{n-1}, a), \right. \\ \left. \frac{1}{2} d(Tx_{n+1}, Tx_{n-1}, a) \right\},$$

and also

$$d(Tx_{n+1}, Tx_n, Tx_{n-1}) = 0.$$

Now, as in (Ram 1982 or Singh-Tiwari-Gupta 1980), it can be shown that  $\{Tx_n\}$  is a Cauchy sequence. Since  $T(X)$  is complete, it has a limit in  $T(X)$ . Call it  $p$ . Then there exists a point  $z$  in  $X$  which is a pre-image of  $p$  under  $T$ , that is  $Tz = p$ .

Now, for any  $n, m$ ,  $n > m$  by (1.1),

$$\begin{aligned} d(S_n x_{n-1}, S_m z, a) &\leq q \cdot \max \left\{ d(Tx_{n-1}, Tz, a), \right. \\ &\quad d(S_n x_{n-1}, Tx_{n-1}, a), \\ &\quad d(S_m z, Tz, a), \\ &\quad \left. \frac{1}{2} [d(S_n x_{n-1}, Tz, a) \right. \\ &\quad \left. + d(S_m z, Tx_{n-1}, a)] \right\} \\ &= q \cdot \max \left\{ d(Tx_{n-1}, Tz, a), 0, \right. \\ &\quad d(S_m z, Tz, a), \\ &\quad \left. \frac{1}{2} [d(Tx_n, Tz, a) \right. \\ &\quad \left. + d(S_m z, Tx_{n-1}, a)] \right\}. \end{aligned}$$

Making  $n \rightarrow \infty$ , we obtain

$$d(Tz, S_m z, a) \leq q \cdot d(S_m z, Tz, a).$$

Since  $a$  is arbitrary,

$$Tz = S_m z.$$

This is true for any  $m$ . Hence  $z$  is a coincidence point of

$T$  and  $S_i$ ,  $i=1, 2, 3, \dots$ .

Now assume that  $T$  and  $S_i$ , for each  $i$ , commute at  $z$  i.e.,

$$TS_i z = S_i Tz, \quad i=1, 2, 3, \dots$$

Also

$$TS_i z = S_i Tz = S_i S_j z = S_i S_j z.$$

Then by (1.1), for  $m \neq n$ ,

$$\begin{aligned} d(S_n z, S_n S_n z, a) &= d(S_n z, S_n S_m z, a) \\ &= d(S_n z, S_m S_n z, a) \\ &\leq q \cdot \max \left\{ d(Tz, TS_n z, a), \right. \\ &\quad d(S_n z, Tz, a), \\ &\quad d(S_m S_n z, TS_n z, a), \\ &\quad \left. \frac{1}{2} [d(S_n z, TS_n z, a) \right. \\ &\quad \left. + d(S_m S_n z, Tz, a)] \right\} \\ &= q \cdot d(S_n z, S_n S_n z, a), \end{aligned}$$

yielding

$$S_n S_n z = S_n z = Tz.$$

So  $Tz$  is a fixed point of  $S_n$  for every  $n$ , naturally. Also, since  $TTz = TS_n z = S_n Tz = S_n S_n z = S_n z = Tz$ ,  $Tz$  is a fixed point of  $T$ . Thus  $Tz$  is a fixed point of  $T$  and the family  $\{S_n\}$ . The uniqueness of the common fixed point follows easily.

**COROLLARY 3.** Let  $T_1, T_2$  and  $T$  be mappings from a 2-metric space  $X$  to itself such that  $T_1(X) \cup T_2(X) \subseteq T(X)$ , and for every  $x, y, a$  in  $X$

$$(3.1) \quad d(T_1 x, T_2 y, a) \leq q \cdot \max \left\{ d(Tx, Ty, a), \right. \\ \left. d(T_1 x, Tx, a), \right.$$

$$\begin{aligned}
& d(T_2y, Ty, a), \\
& \frac{1}{2}[d(T_1x, Ty, a) \\
& + d(T_2y, Tx, a)] \Big\}.
\end{aligned}$$

If  $T(X)$  is a complete subspace of  $X$ , then  $T_1$ ,  $T_2$  and  $T$  have a coincidence point  $z$ . Further if  $T$  commutes with each of  $T_1$  and  $T_2$  at  $z$  then  $T$ ,  $T_1$  and  $T_2$  have a common unique fixed point. Indeed  $Tz$  is the unique common fixed point.

PROOF. The consequences are immediate if one takes  $\{S_n\} = \{T_1, T_2, T_1, T_2, T_1, \dots\}$  in Theorem 2.

REMARK 4. Corollary 3 improves a number of fixed point theorems for two and three mappings on 1-metric and 2-metric spaces (See, for instance, Singh-Tiwari-Gupta 1980, Singh 1982, Singh-Pant 1983, Singh-Mishra 1983).

As a variant of Theorem 2, we have the following:

THEOREM 5. Let  $\{S_n\}$  be a sequence of mappings from a 2-metric space  $X$  to itself. Let  $T$  be a mapping from  $X$  to  $X$  satisfying (2.1) and (2.2). If there is a non-negative integer  $m_i$  for each  $S_i$  such that for all  $x, y, a$  of  $X$  and for every pair  $i, j$  with  $i \neq j$ ,

$$\begin{aligned}
& d(S_i^{m_i}x, S_j^{m_j}y, a) \leq q \cdot \max \{ d(Tx, Ty, a), \\
& d(S_i^{m_i}x, Tx, a), \\
& d(S_j^{m_j}y, Ty, a), \\
& \frac{1}{2}[d(S_i^{m_i}x, Ty, a) \\
& + d(S_j^{m_j}y, Tx, a)] \}
\end{aligned}$$

Then  $T$  and  $S_i^{m_i}$  ( $i=1, 2, 3, \dots$ ) have a coincidence point  $z$ . Further if  $T$  and  $S_i^{m_i}$  ( $i=1, 2, 3, \dots$ ) commute at  $z$  then  $T$  and the sequence  $\{S_n\}$  of mappings have a unique common fixed point.

PROOF. Clearly  $S_i^{m_i}(X) \subseteq S_i(X) \subseteq T(X)$ . Thus Theorem 2 pertains to  $S_i^{m_i}$  and  $T$ . So there is a unique point  $p$  in  $T(X)$  such that

$$p = Tz = S_i^{m_i}z$$

and

$$S_i^{m_i}p = Tp = p, \quad i=1, 2, 3, \dots.$$

From this

$$S_i p = S_i S_i^{m_i} p = S_i^{m_i} S_i p.$$

This shows that  $S_i p$  is a fixed point of  $T$  and  $S_i^{m_i}$ . The uniqueness of  $p$  implies that

$$p = Sp = S_i p.$$

REMARK 6. Results of Lal-Singh(1978), Rhoades(1979), Ram(1982) and Singh-Tiwari-Gupta(1980) may be obtained as corollaries. In particular, an improved version of Corollary 1 of Lal-Singh(1978) is obtained when  $T$  is an identity mapping in the above theorem.

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