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CERTAIN SUBCLASSES OF ANALYTIC P-VALENT FUNCTIONS

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1. Introduction

Let A_p denote the class of functions of the form

$$f(z)=z^p+\sum_{k=1}^{\infty} a_{p+k}\, z^{p+k} \quad (p\in N=\{1,2,\cdots\}) \tag{1.1}$$

which are analytic in the unit disk $U=\{z:|z|<1\}$. For $f(z)$ and $g(z)$ being in the class A_p, $f(z)$ is said to be subordinate to $g(z)$ if there exists a Schwarz function $w(z)$, $w(0)=0$ and $|w(z)|<1$ for $z\in U$, such that $f(z)=g(w(z))$. We denote by $f(z)\prec g(z)$ this relation. In particular, if $g(z)$ is univalent in the unit disk U, then the subordination $f(z)\prec g(z)$ is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$.

A function $f(z)$ belonging to A_p is said to be in the class $S^*_p\,[a,b]$ if it satisfies

$$\frac{zf'(z)}{pf(z)}\prec\frac{1+az}{1+bz} \tag{1.2}$$

for some a and b with $-1\leqq b<a\leqq 1$, and for all $z\in U$.

Further, a function $f(z)$ belonging to A_p is said to be in the class $K_p[a,b]$ if it satisfies $zf'(z)/p\in S^*_p[a,b]$.

The class $S^*_1[a,b]$ was introduced by Goel and Mehrok ([1], [2]), and Janowski [3]. Further, the class $K_1[a,b]$

was introduced by Silverman and Silvia [5].

Let $S^*_p(a,b)$ be the subclass of A_p consisting of functions which satisfy the condition

$$\left|\frac{zf'(z)}{pf(z)}-a\right|<b \tag{1.3}$$

for some a and b with $a\geqq b$, and for all $z\in U$. Furthermore we denote by $K_p(a,b)$ the subclass of A_p consisting of functions which satisfy the condition $zf'(z)/p\in S^*_p(a,b)$.

The classes $S^*_1(a,b)$ and $K_1(a,b)$ were introduced by Silverman [4] and Silverman and Silvia [5], respectively.

2. Some Properties

We begin with the statement and the proof of the following result.

THEOREM 1. If $-1<b<a\leqq 1$, then

$$S^*_p[a,b]\equiv S^*_p\left(\frac{1-ab}{1-b^2},\ \frac{a-b}{1-b^2}\right). \tag{2.1}$$

Further, if $a\geqq b$, then

$$S^*_p(a,b)\equiv S^*_p\left[\frac{b^2-a^2+a}{b},\ \frac{1-a}{b}\right]. \tag{2.2}$$

PROOF. We employ the same manner by Silverman and Silvia [5]. Let $f(z)\in S^*_p[a,b]$ with $-1<b<a\leqq 1$, that is,

$$\frac{zf'(z)}{pf(z)}\prec F(z)=\frac{1+az}{1+bz}. \tag{2.3}$$

By using the result due to Singh and Goel [6], we have

$$\left|F(z)-\frac{1-ab|z|^2}{1-b^2|z|^2}\right|\leqq\frac{(a-b)|z|}{1-b^2|z|^2}\quad(z\in U). \tag{2.4}$$

It follows from (2.4) that $F(z)$ maps the circle $|z|=1$ onto a circle

$$\left|w-\frac{1-ab}{1-b^2}\right|=\frac{a-b}{1-b^2}. \tag{2.5}$$

This implies that

$$\left|\frac{zf'(z)}{pf(z)}-\frac{1-ab}{1-b^2}\right|<\frac{a-b}{1-b^2}\,(-1<b<a\leqq 1;\ z\in U). \tag{2.6}$$

Noting $(1-ab)/(1-b^2)\geqq(a-b)/(1-b^2)$ for $-1<b<a\leqq 1$, we obtain (2.1).

Next, let $f(z)\in S^*_p(a,b)$ with $a\geqq b$, that is,

$$\left|\frac{zf'(z)}{pf(z)}-a\right|<b. \tag{2.7}$$

Then we have to find A and B with $-1<B<A\leqq 1$ such that $a=(1-AB)/(1-B^2)$ and $b=(A-B)/(1-B^2)$ for $a\geqq b$. Because we find such A and B, then we have

$$\left|\frac{zf'(z)}{pf(z)}-\frac{1-AB|z|^2}{1-B^2|z|^2}\right|\leqq\frac{(A-B)|z|}{1-B^2|z|^2} \tag{2.8}$$

which implies

$$\frac{zf'(z)}{pf(z)}\prec\frac{1+Az}{1+Bz}, \tag{2.9}$$

or $f(z)\in S^*_p[A,B]$.

Letting $A=B+b(1-B^2)$ for $-1<B<A\leqq 1$, we obtain

$$bB^3+(a-1)B^2-bB+1-a=0. \tag{2.10}$$

The above equation (2.10) has the solutions $B=\pm 1$, $(1-b)/a$. Since $-1<B<A\leqq 1$, we only take $B=(1-b)/a$. Thus we have $B=(1-b)/a$ and $A=(b^2-a^2+a)/b$. This completes the proof of (2.2).

THEOREM 2. If $-1<b<a\leqq 1$, then

$$(2.11)\qquad K_p[a, b] \equiv K_p\left(\frac{1-ab}{1-b^2},\ \frac{a-b}{1-b^2}\right).$$

Further, if $a \geqq b$, then

$$(2.12)\qquad K_p(a, b) \equiv K_p\left[\frac{b^2-a^2+a}{b},\ \frac{1-a}{b}\right].$$

PROOF. Note that $f(z) \in K_p[a, b]$ if and only if $zf'(z)/p \in S^*_p[a, b]$, and that $f(z) \in K_p(a, b)$ if and only if $zf'(z)/p \in S^*_p(a, b)$. Therefore the proof of Theorem 2 follows form Theorem 1.

Next we prove

THEOREM 3. $S^*_p(a_1, b_1) \subseteq S^*_p(a_2, b_2)$ if and only if $|a_2-a_1| \leqq b_2-b_1$. Furthermore, $S^*_p[a_1, b_1] \subseteq S^*_p[a_2, b_2]$ if and only if $|a_2b_1-a_1b_2| \leqq (a_2-a_1)-(b_2-b_1)$.

PROOF. Since $S^*_p(a_1, b_1) \subseteq S^*_p(a_2, b_2)$ if and only if

$$\{w : |w-a_1| < b_1\} \subseteq \{w : |w-a_2| < b_2\},$$

or, if and only if $a_2-b_2 \leqq a_1-b_1$ and $a_1+b_1 \leqq a_2+b_2$, we have $S^*_p(a_1, b_1) \subseteq S^*_p(a_2, b_2)$ if and only if $|a_2-a_1| \leqq b_2-b_1$. In view of Theorem 1, we note that $S^*_p[a_1, b_1] \subseteq S^*_p[a_2, b_2]$ if and only if

$$\{w : |w-A_1| < B_1\} \subseteq \{w : |w-A_2| < B_2\},$$

where $A_j = (1-a_jb_j)/(1-b_j^2)$ and $B_j = (a_j-b_j)/(1-b_j^2)$, which equivalent to $|A_2-A_1| \leqq B_2-B_1$. Thus we have $S^*_p[a_1, b_1] \subseteq S^*_p[a_2, b_2]$ if and only if $|a_2b_1-a_1b_2| \leqq (a_2-a_1)-(b_2-b_1)$.

Finally, we have

THEOREM 4. $K_p(a_1, b_1) \subseteq K_p(a_2, b_2)$ if and only if $|a_2-a_1| \leqq b_2-b_1$. Furthermore, $K_p[a_1, b_1] \subseteq K_p[a_2, b_2]$ if and only if $|a_2b_1-a_1b_2| \leqq (a_2-a_1)-(b_2-b_1)$.

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