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ON FINITELY GENERATED SEMIPRIME ALGEBRA OVER COMMUTATIVE RINGS

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1. Introduction

Let R be a commutative ring. E.P. Armendariz studied in [2] that a semiprime finitely generated R-algebra when R is a regular ring and that by combining the above fact with the results of [6, 7], R is semiprime and every $f.g.$ semiprime R-algebra A is Azumaya if the ring R is regular.

In this paper, we prove converse of Armendariz's theorem and we get a necessary and sufficient condition on which a regular ring R is π-regular.

That is, we have the following results;

1) Let R be a commutative ring. Then the following are equivalent;

i) R is von Neumann regular.

ii) R is semiprime and every $f.g.$ semiprime R-algebra is Azumaya.

2) Let R be a commutative ring. Then the following are equivalent;

i) R is von Neumann regular.

ii) Every integral extension of R is π-regular.

An algebra A is called *Azumaya* if R is both central and separable. The ring R is said to be *P.I. ring* if R satisfies a polynomial identity with coefficients in the centroid and at least one coefficient is invertible. All other notations and terminologies will follow from [2] and [4].

2. Preliminaries

Kaplansky made the following conjecture in [4]: A ring R is von Neumann regular if and only if R is semiprime and each prime factor ring of R is von Neumann regular. That the conjecture fails to hold in general was shown by a counter example of J.W. Fisher and R.L. Snider.

THEOREM 2.1 [4]. A ring R is von Neumann regular if and only if R is semiprime, the union of any chain of semiprime ideals of R is a semiprime ideal of R and each prime factor ring of R is von Neumann regular.

Since any finitely generated algebra over a commutative ring satisfies a polynomial identity (is a P.I.-algebra), this leads to consideration of semiprime P.I.-algebra with regular center.

THEOREM 2.2 [2]. Let A be a semiprime finitely generated algebra over a commutative regular ring R. Then A is a regular ring.

The ring R is finitely generated as a ring over its center $Z(R)$, if R is an epimorphic image of a free (non commutative) ring over $Z(R)$ generated by finitely many indeterminates $[x_1, x_2, \cdots, x_n]$ which only commute with elements of $Z(R)$. Following C. Proces, the ring R is called an *affine ring* if R is finitely generated over its center $Z(R)$.

THEOREM 2.3 [7]. Let R be an affine ring. Then the following properties are equivalent;

1) Every simple right R-module is injective.
2) R is von Neumann regular.
3) R is biregular.

THEOREM 2.4[2]. Let A be an algebra over a regular ring with center of A being R. A is Azumaya over R if and only if A is a biregular ring which is finitely generated over R.

Combining Theorems 2.2, 2.3 and 2.4, we have the following result.

THEOREM 2.5 [2]. Let A be a finitely algebra over a ~~regular ring. The following conditions on A are equivalent~~:

1) A is semiprime.
2) A is regular.
3) A is biregular.
4) A is semiprime Azumaya algebra.

The follwing theorem was shown by Storrer.

THEOREM 2.6 [4]. Let R be a P.I. ring. Then the following are equivalent:

1) R is π-regular.
2) Each prime ideal of R is primitive.
3) Each prime ideal of R is maximal.
4) R is left (right) π-regular.
5) $R/rad(R)$ is π-regular, where $rad(R)$ is prime radical.
6) Each prime factor ring of R is von Neumann regular.

3. Main results

LEMMA 3.1. Let R be a commutative prime ring and $0\neq a$

$\in R$. If $A=\begin{pmatrix} R & aR \\ aR & R \end{pmatrix}$ is Azumaya, then a is invertible in R.

PROOF. It is easily checked that R coincides with the center $Z(A)$. Now if A is Azumaya, $A\otimes_R A^{op}\cong Hom_R(A, A)$. In this case $\sigma(a\otimes b)(x)=axb$ for $x\in A$.

Consider $f\in Hom_R(A, A)$ such that $f(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix})=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Then since A is Azumaya, there are $\begin{pmatrix} x_i & ay_i \\ az_i & w_i \end{pmatrix}$ and $\begin{pmatrix} x_i' & ay_i' \\ az_i' & w_i' \end{pmatrix}$ in A, $1\leqq i\leqq n$ for some n such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}=\sum_{i=1}^{n}\begin{pmatrix} x_i & ay_i \\ az_i & w_i \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_i' & ay_i' \\ az_i' & w_i' \end{pmatrix}.$$

By this relation, we have $1\in a^2R$ and so a is invertible in R.

THEOREM 3.2. Let R be a commutative ring. Then the following are equivalent.

1) R is von Neumann regular.

2) R is semiprime and every $f.g.$ semiprime R-algebra A is Azumaya.

PROOF. Assume that R is von Neumann regular. By Theorem 2.5, A is Azumaya algebra.

For the opposite direction, let P be a prime ideal of R. We will show that P is a maximal ideal of R. Now, take $a\in R$ and consider an R-algebra $A=\begin{pmatrix} R & aR \\ aR & R \end{pmatrix}$. Then A is a finitely generated semiprime algebra over R. In this case, the center $Z(A)=\left\{\begin{pmatrix} x & 0 \\ 0 & w \end{pmatrix} \mid (x-w)a=0\right\}$. By our assumption, A is separable over $Z(A)$.

Consider the mapping $\sigma: A \longrightarrow \begin{pmatrix} R/P & aR/P \\ aR/P & R/P \end{pmatrix}$ with $\sigma\left[\begin{pmatrix} x & ay \\ az & w \end{pmatrix}\right]$ $= \begin{pmatrix} x+p & ay+p \\ az+p & w+p \end{pmatrix}$, where $\bar{a}=a+P$. Then since $a \notin P$, we have that $Ker\ \sigma = \left\{\begin{pmatrix} x & ay \\ az & w \end{pmatrix} \middle| x, y, z, w \in P\right\} = PA$. Therefore $A/PA \cong \begin{pmatrix} R/P & \bar{a}R/P \\ \bar{a}R/P & R/P \end{pmatrix}$.

Now since $PZ(A)A = PA$ and A is Azumaya, we have $PA \cap Z(A) = PZ(A)$. So A/PA is Azumaya over $Z(A)/PZ(A)$. Also in this case $Z(A/PA) = Z(A)/PZ(A)$ [1]. But since $A/PA \cong \begin{pmatrix} R/P & \bar{a}R/P \\ \bar{a}R/P & R/P \end{pmatrix}$, we have $Z(A/PA) \cong R/P$. So $\begin{pmatrix} R/P & \bar{a}R/P \\ \bar{a}R/P & R/P \end{pmatrix}$ is Azumaya over R/P. Therefore, by our Lemma 3.1, $\bar{a}$ is invertible in R/P. Hence R/P is a field. Thus R is a von Neumann regular ring.

COROLLARY 3.3. Let R be a commutative ring, then the following are equivalent:

1) R is von Neumann regular.

2) R is semiprime and for every finitely generated R-algebra A, $J(A)$ is nilpotent and $A/J(A)$ is Azumaya.

PROOF. In [2], E. P. Armendariz proved that if R is von Neumann regular then $J(A)$ is nilpotent and $A/J(A)$ is a regular ring.

Conversely, let P be a prime ideal and $a \notin P$. Then $A = \begin{pmatrix} R & aR \\ aR & R \end{pmatrix}$ is finitely generated semiprime R-regular. But since A is a normalizing finite extension of R, we have

$0=J(R)=R\cap J(A)$ and so $R\cong A/J(A)$. This shows that $A/J(A)$ is R-algebra.

Now since A is semiprime and $J(A)$ is nilpotent, $J(A)=0$. Therefore A is Azumaya. By Theorem 3.2, R is von Neumann regular.

Let A be a ring with identity. Consider the condition (*) the ring A satisfies a polynomial identity $f(x_1, x_2,\cdots,x_n)=0$ for which f has coefficient in C, the center of A, and for which at each prime ideal P of A, f induces a nontrivial polynomial identity on A/P.

THEOREM 3.4 [5]. Let A be a ring with identity which is integral over unital subring B of C, the center of A, suppose futher that B satisfies (*), then; If P is prime ideal of A, P is maximal ideal of A if and only if $P\cap A$ is maximal ideal of B.

THEOREM 3.5. Let R be a commutative ring. Then the following are equivalent;

1) R is von Neumann regular.
2) Every integral extension of R is π-regular.

PROOF. Suppose that R is von Neumann regular and A is integral extension of R. To show that A is π-regular, let P be a prime ideal of A. Then A/P is integral over $R/P\cap R$. Since P is a maximal ideal of A, $P\cap R$ is maximal ideal of R. Therefore $R/P\cap R$ is a field. By Theorem 2.6, A/P is π-regular. Thus A is π-regular. Conversely, since $A=\begin{pmatrix} R & R \\ R & R \end{pmatrix}$ is integral extension of R, it is π-regular. It follows that R is von Neumann regular.

References

[1] A. Brown, On Artin's theorem and Azumaya algebras, J. Algebra, 77(1982) 323-332.

[2] E. P. Armendariz, On semiprime P. I. -algebra over commutative regular rings, Pacific. J. Math., 66(1976) 23-28.

[3] F. DeMeyer and E. C. Ingraham, Separable algebras over commutative rings, Lecture Notes in Mathematics, Vol. 181, Springer-Verlag, New York and Berlin 1971.

[4] J. W. Fisher and R. L. Snider, On the von Neumann regular prime factor rings, Pacific. J. Math., 54(1974) 135-144.

[5] A. G. Heinecke, A remark about noncommutative integral extensions, Canad. Math. Bull., 13(1970) 359-361.

[6] G. Michler and O. Villamayor, On rings whose simple modules are injective, J. Algebra, 25(1973) 185-201.

[7] J. A. Wehlen, Algebras over absolutely flat commutative rings, Trans. Amer. Math. Soc., 196(1974) 149-160.

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