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Differential Game Approach to Competitive Advertising Model

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Abstract

This paper presents an adaptive algorithm to generate a near-optimal closed-loop solution for a non-zero sum differential game by periodically updating the solutions of the two-point boundary-value problem.

Applications to competitive advertising problem show that the adaptive algorithm can be used as an efficient tool to solve the differential game problem in which one player may take advantage of the other's non-optimal play.

1. Introduction

Deal and Zionts [8, 9] derived the open-loop control for the case in which two players always choose their optimal strategies against each other. However, it has been pointed out that the closed-loop control laws are needed to develop a generalized differential game model in which one player can take advantage of the competitor's non-optimal play [2, 7, 10].

Anderson [2] developed a robust technique to derive the near-optimal solution which can be applied to realistic non-linear differential game problem. The purpose of this paper is to improve Anderson's method so that it can be applied to the analysis of the market behaviors of a new-entering company (PURSUER) and its target company (EVADER), i.e., pursuit-evasion game.

2. Differential Game

(1) Overview

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The study of differential game was initiated by Issacs [11]. Berkowitz and Flemming [5] applied rigorous classical variational techniques to simple differential games and Berkowitz [6] expanded the applicable class of problems. Typical structure of the differential game problem can be briefly described as follows [10] : Determine a saddle point for

$$J = \phi[x(t_f), t_f] + \int_{t_0}^{t_f} L(x,u,v,t)dt \tag{1}$$

subject to the constraints

$$x' = f(x, u, v, t)$$

$$x(t_0) = x_0, \tag{2}$$

$$\psi\ [x(t_f), t_f] = 0$$

$$\text{and } u\epsilon U(t),\ v\epsilon V(t) \tag{3}$$

where J is the payoff, x and x' are the (vector) position or state of the game and its derivative, t_f and t_0 are terminal time and initial time, u and v are piecewise continuous vector functions called strategies, $\psi(.)$ is an algebraic terminal condition. A saddle point is defined as the pair (u^*, v^*) satisfying the relation

$$J(u^*, v) \geqq J(u^*, v^*) \geqq J(u, v^*) \tag{4}$$

for arbitrary $u\epsilon U$, $v\epsilon V$. If equation (4) can be realized, u^* and v^* are called optimal pure strategies and $J(u^*, v^*)$ is called the value of the game or performance value. Differential game problem is always a minimax case. Control vectors u and v try to minimize and maximize the value of the game, respectively. The vector f and the scalar L are both assumed to be separable in u and v (separability condition) which are also assumed to be independent of state vector x.

(2) Pursuit-evasion game

In the structure of the pursuit-evasion game, the maximizing player is generally an "evader" and the minimizing player a "pursuer". Note that in the context of the pursuit-evasion game, the pursuer should always have the "pursuit mechanism", by which the pursuer can pursue and even capture the evader.

In engineering problem, the pursuer's system should be "more controllable" than the evader's. The pursuit mechanism in management applications is not easy to define and can be termed as a "strategic market planning" [1].

3. An Adaptive Algorithm

(1) Statement of the problem

Consider the equations for differential game (1) through (3). Suppose that terminal time t_f is specified. The Hamiltonian is

$$H = L(x, u, v, t) + \lambda^T f(x, u, v, t) \tag{5}$$

where λ is an n-dimensional costate vector. The costate equations and transversality conditions on $\lambda(t_f)$ are

$$\lambda'^T = -H_x(x, u, v, t),$$
$$\lambda(t_f) = (\phi_x + \nu \psi_x)\Big|_{t_f}$$

where ν is a scalar Lagrange multiplier and H_x is the derivative of H with respect to x.

The optimal control for the pursuer, $u^*(x,\lambda,t)$ is found by minimizing H with respect to u, while the evader's optimal control $v^*(x,\lambda,t)$ is found by maximizing H with respect to v. Substituting these expressions for $u^*(x,\lambda,t)$ and $v^*(x,\lambda,t)$ into the state and costate differential equations, the following TPBVP (Two-Point Boundary-Value Problem) is obtained:

$$\begin{aligned} x' &= f(x, \lambda, t) \\ \lambda' &= g(x, \lambda, t) \\ x(t_o) &= x_o \\ \lambda(t_f) &= (\phi_x + \nu\psi_x)\Big|_{t_f} \\ \psi[x(t_f), t_f] &= 0 \end{aligned} \tag{6}$$

(2) Algorithm Formulation

Suppose that at time t_1, a deviation δx between the actual state and the reference state calculated from the TPBVP solution emerges from evader's non-optimal play.

$\delta x(t_1)$ gives rise to a change in the costate, i.e., $\delta\lambda(t_1)$. To find the resulting $\delta\lambda(t_1)$ as a function of $\delta x(t_1)$, consider the linearized state and costate equations

$$x' = f_x\,\delta x + f_\lambda \delta\lambda \tag{7}$$
$$\lambda' = g_x\,\delta x + g_\lambda \delta\lambda \tag{8}$$

Also $\delta x(t_1)$ invokes $\delta x(t_f)$, $\delta\lambda(t_f)$, and $d\nu$.
Linearizing the conditions at t_f in the TPBVP and using dt_f=0, we obtain

$$\delta\lambda(t_f) = \phi_{xx}\Big|_{t_f}\,\delta x(t_f) \tag{9}$$

When terminal time is either free or fixed, $\delta\lambda(t_f)$ can always be expressed as a function of $\delta x(t_f)$ in the form

$$\delta\lambda(t_f) = A\,\delta x(t_f) \tag{10}$$

where A is n x n matrix.
Then the solution for equations (7) and (8) can be written as

$$\delta x(t) = \Phi_{xx}(t, t_f)\,\delta x(t_f) + \Phi_{x\lambda}(t, t_f)\,\delta\lambda(t_f)$$

$$= [\Phi_{XX}(t, t_f) + \Phi_{X\lambda}(t, t_f)A] \; \delta x(t_f) \tag{11}$$

$$\delta\lambda(t) = \Phi_{\lambda X}(t, t_f)\, \delta x(t_f) + \Phi_{\lambda\lambda}(t, t_f)\, \delta\lambda(t_f)$$

$$= [\Phi_{\lambda X}(t, t_f) + \Phi_{\lambda\lambda}(t, t_f)A] \; \delta x(t_f) \tag{12}$$

where Φ matrices are fundamental matrices [12].
Using equations (11) and (12), $\delta\lambda(t_1)$ can be obtained as a function of $\delta x(t_1)$, that is,

$$\delta\lambda(t_1) = [\Phi_{\lambda X}(t_1, t_f) + \Phi_{\lambda\lambda}(t_1, t_f)A] \; [\Phi_{XX}(t_1, t_f) + \Phi_{X\lambda}(t_1, t_f)A]^{-1} \; \delta x(t_1) \tag{13}$$

After the TPBVP solution is updated at t_1 using equation (13), a forward integration of the updated solution to the final conditions usually results in terminal conditions that do not satisfy the transversality conditions. With these new end conditions, the TPBVP and fundamental matrices should be integrated backward from t_f to the next updating time. This then yields all the information required to obtain a new optimal reference solution which is used to update the TPBVP solution.

The procedures discussed above constitute a "near-optimal" method.

4. Applications to Competitive Advertising Model

(1) Competitive Advertising Model

Deal's advertising model [8] is taken as a prototype example:

$$\max_{u_1} J_1 = w_1 x_1(t_f)/[x_1(t_f) + x_2(t_f)] + \int_{t_0}^{t_f} [c_1 x_1(t) - u_1^2(t)]\, dt$$

$$\max_{u_2} J_2 = w_2 x_2(t_f)/[x_1(t_f) + x_2(t_f)] + \int_{t_0}^{t_f} [c_2 x_2(t) - u_2^2(t)]\, dt$$

subject to

$$x'_1(t) = -a_1 x_1(t) + b_1 u_1(t)\, [M - x_1(t) - x_2(t)]/M$$

$$x'_2(t) = -a_2 x_2(t) + b_2 u_2(t)\, [M - x_1(t) - x_2(t)]/M,$$

$x_1(t_0)$, $x_2(t_0)$ given,

$u_1(t), u_2(t), x_1(t), x_2(t) > 0.$

where $x_i(t)$ is the sales rate of firm i at time t, $x'(t)$ is the first derivative with respect to time, $a_i > 0$ is a decay parameter, $b_i > 0$ is an advertising effectiveness parameter, $u_i(t)$ is the rate of advertising expenditures of firm i at time t, $w_i(t)$ is the weighting factor for the performance index, and M is the total potential market size.

It is assumed that total potential market size is fixed at M of sales and that advertising affects sales but sales do not directly determine the level of advertising.

Integral form in performance indices shows that the advertising media is assumed to be relatively continuous such as radio, television, and newspapers. Also, performance index indicates that a weighting factor, w_i, is allowed to vary from total profit orientation to terminal market share orientation cases. System dynamic equations represent that there is a recognition of the competition's effect and that each of the market competitors has essentially the same form for its dynamics and that the sales effectiveness of succeeding advertising expenditures diminishes as both firm's sales increase.

(2) Computational Results and Implications

Numerical algorithm presented in section 3 is applied to the competitive advertising model above. Flowchart of the solution procedure is shown in Figure 1.

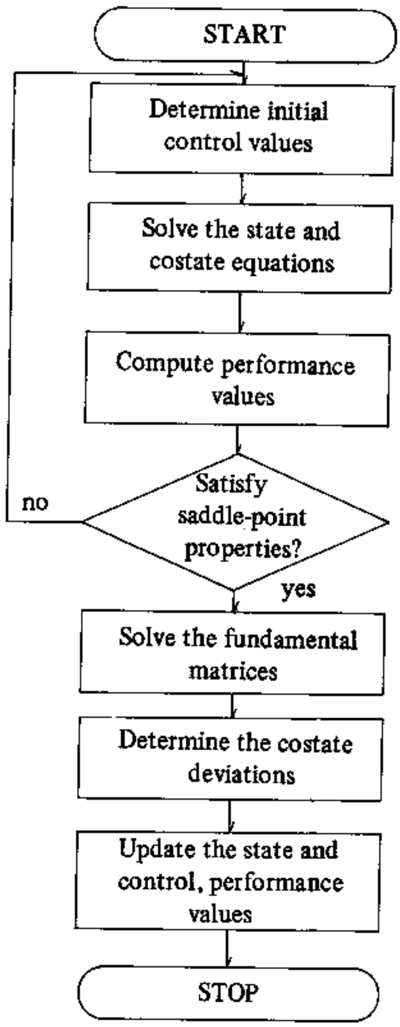

Figure 1. Flowchart of the solution procedure

Parameters of the prototype example are assumed to be given as

$a_1 = 0.20$	$a_2 = 0.25$
$b_1 = 1.10$	$b_2 = 1.10$
$c_1 = 0.60$	$c_2 = 0.80$
$w_1 = 15.0$	$w_2 = 8.00$
$x_1(t_0) = 40.0$	$x_2(t_0) = 100.0$
$u_1(t) = 1.00$	$u_2(t) = 1.00$

where pursuer and evader are denoted as 1 and 2, respectively. M is given 500.0 and $t \epsilon [0.0, 5.0]$ and time is discretized into 1.0 for analytical simplicity. Each of the firms involved in the game is assumed to have perfect knowledge of the model parameters. From the parameter values and initial values, it is easily verified that the pursuer has a strong desire to raise the current market share up to the evader's at the terminal planning time (note that the pursuer's weighting constant for the relative terminal market share, w_1, is greater than the evader's w_2) and that the pursuer is supposed to advertise effectively (note that pursuer's decay parameter a_1 is smaller than the evader's a_2). Table shows the state deviations observed.

Table 1. State Deviations Observed

Time	Pursuer	Evader
1.0	–1.0	–2.0
2.0	–0.5	–1.5
3.0	0.0	–1.0
4.0	0.5	–0.5

The computational results are summarized in Tables 2 and 3. Figures 2 and 3 represent the optimal trajectories of the reference solutions calculated from TPBVP and the updated state values, respectively.

Table 2. Computational Results for Prototype Example

	Pursuer		Evader	
	Before*	After**	Before	After
performance value :	180.86	238.72	692.89	983.90
terminal market share (%) :	7.80	13.95	19.80	41.10

*: before applying an adaptive algorithm

**: after applying an adaptive algorithm

Table 3. Deviations of Advertising Expenditures After Applying The Adaptive Algorithm

Time	Pursuer	Evader
2.0	.8571E-02	.9665E-02
3.0	.4672E-02	.5464E-02
4.0	.1540E-02	.1867E-02
5.0	–.4484E-04	.2166E-04

Table 2 represents that the pursuer's terminal market share increases by the amount of 12.0% when the algorithm is applied. It can be observed in Figure 2 that at earlier planning period, the effects of advertising expenditures, have a great impact on the state values to yield higher sales level and diminish as time reaches to t_f.

Figure 3 shows that updated reference value x_1 and x_2 at time 3.0 are very large as compared with the original reference values. This results from the most negative deviation of state values at time 1.0 (see Table 1) which causes the greatest increase of advertising expenditures at time 2.0 (see Table 3) to make the sales level at time 3.0 higher than any other state values. Tables 4, 5 and 6 illustrate the computational results for various parameter values. Tables 5 and 6 show that pursuer can improve his advertising policy during differential game to obtain higher performance value and terminal market share when using the adaptive algorithm as a tool to build up an advertising strategy which could be taken as an useful pursuit mechanism.

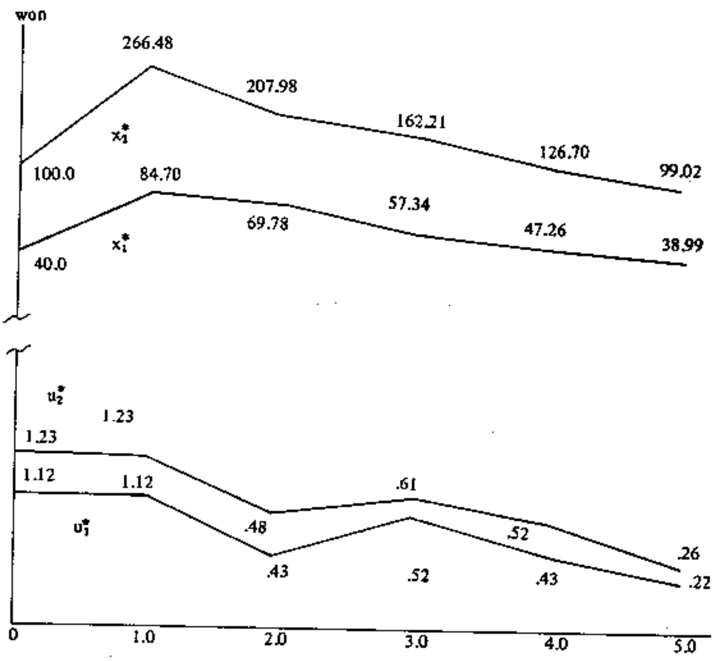

Figure 2. Optimal Trajectories of Reference Solution

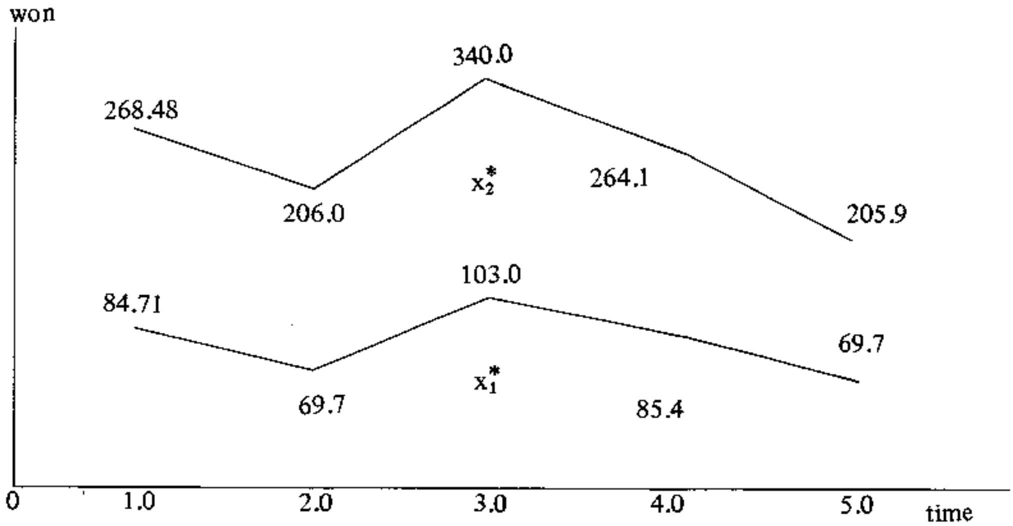

Figure 3. Optimal Trajectories of Updated State Values

Table 4. Parameter Values and Initial State Values

Case	a_1	a_2	b_1	b_2	c_1	c_2	$x_1(0)$	$x_2(0)$	w_1	w_2	M
1	0.25	0.25	2.00	1.50	0.50	0.60	40.0	100.0	22.5	8.0	500.0
2	0.25	0.25	2.00	1.50	0.50	0.60	40.0	100.0	25.0	5.0	500.0
3	0.25	0.25	2.00	1.50	0.40	0.60	40.0	100.0	75.9	9.0	500.0
4	0.25	0.25	1.10	1.10	0.50	0.60	40.0	100.0	7.5	8.0	500.0
5	0.25	0.25	1.10	1.10	0.50	0.60	40.0	100.0	2.5	8.0	500.0
6	0.20	0.25	1.10	1.10	0.50	0.60	40.0	100.0	15.0	8.0	500.0
7	0.15	0.25	1.10	1.10	0.50	0.60	40.0	100.0	15.0	8.0	500.0
8	0.25	0.40	1.10	1.10	0.50	0.60	40.0	100.0	15.0	8.0	500.0
9	0.35	0.50	1.10	1.10	0.50	0.60	40.0	100.0	15.0	8.0	500.0
10	0.20	0.25	1.10	1.10	0.60	0.80	40.0	100.0	15.0	8.0	500.0
11	0.10	0.25	1.10	1.10	0.20	0.80	25.0	25.0	12.0	8.0	400.0
12	0.10	0.25	1.10	1.10	0.20	0.80	25.0	25.0	9.0	6.0	300.0
13	0.10	0.25	1.10	1.10	0.20	0.80	25.0	25.0	6.0	4.0	200.0
14	0.10	0.25	1.10	1.10	0.20	0.80	30.0	70.0	6.0	4.0	200.0

Table 5. Comparison of Performance Values

Case	Pursuer		Evader	
	Before	After	Before	After
1	157.90	214.90	517.06	701.65
2	164.08	221.15	514.89	699.48
3	192.19	250.53	410.51	555.13
4	272.97	234.12	523.88	712.25
5	170.56	232.70	523.88	712.25
6	152.77	194.10	523.77	712.00
7	133.70	159.04	523.68	710.77
8	174.67	237.30	779.89	1287.29
9	227.48	350.85	1035.45	1935.57
10	180.86	238.72	692.89	983.90
11	33.50	36.33	155.05	199.32
12	32.49	35.37	156.80	202.89
13	31.67	34.81	160.75	210.08
14	38.07	42.45	488.25	661.72

Table 6. Comparison of Terminal Market Shares

Case	Pursuer		Evader	
	Before	After	Before	After
1	7.60(%)	15.42(%)	19.70(%)	40.83(%)
2	7.60	15.42	19.70	40.83
3	9.70	19.86	15.70	32.28
4	7.90	16.46	19.85	41.30
5	7.90	16.46	19.84	41.30
6	7.80	14.04	19.85	41.29
7	7.70	11.92	19.86	41.28
8	7.80	16.49	19.91	65.52
9	7.90	22.38	19.93	88.74
10	7.80	13.90	19.80	41.10
11	6.01	7.73	6.01	10.96
12	8.07	10.42	8.03	14.87
13	12.17	15.94	12.07	23.01
14	14.82	19.94	34.72	72.10

(3) Comparison with Anderson's Method

Computational results obtained by applying Anderson's solution procedure [2] to the prototype example mentioned above are summarized in Table 7.

Table 7. Comparative Results for Prototype Example

	Pursuer		Evader	
	After	Anderson*	After	Anderson
performance value :	238.72	229.94	983.90	941.07
terminal market share(%) :	13.95	13.94	41.10	41.10

* : applying an Anderson's method

Table 7 shows that the adaptive algorithm proposed in this paper gives higher performance value than Anderson's method and yields almost the same terminal market share as Anderson's method. Simulation experiments indicate that for the case 3, 10 in Table 4, the adaptive algorithm yields more efficient strategy and higher performance value to the pursuer than Anderson's method and shows the same results as Anderson's method for the other cases in Table 4.

5. Concluding Remarks

This paper extended the applicability of differential game to a marketing decision problem under the duopolistic competition by developing an adaptive algorithm for updating the solutions of TPBVP. The algorithm turned out to be efficient for establishing the advertising policy of a new-entering company, thereby ensuring higher performance value and terminal market share.

The algorithm can also be used in selecting the optimal combination of marketing strategies such as pricing, market segmentation, product mix, and entry timing.

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