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An Integrated Inventory Model for Two-Product Single-Facility System

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Abstract

In this paper, an integrated production inventory model is developed for two products on a single facility in which the raw materials are fed to the production system as required at the facility. The aim is to determine simultaneously the optimal production schedule for two products and the optimal procurement policies for the raw materials in a way that minimizes the total variable cost of the production system. The production schedule for the more frequently set-up product is formulated by the method of equal or unequal lot sizes.

For the model developed, we present a computational scheme of finding the optimal policies and carry out sensitivity analysis through example problem.

1. Introduction

Many research works [1,2,4,6,7] were undertaken to solve the problem of scheduling lot sizes for several products on a single production facility, called by the "Economic Lot Scheduling Problem (ELSP). The ESLP arises from the desire to accomodate the cyclical production patterns that are based on economic production quanitity (EPQ) calculations for individual products on a single facility. An excellent analysis of these methods can be found in survey paper by Elmaghraby [2]. These works proposed methods for generating feasible schedule with lower cost than the one obtained assuming a common cycle length. However, all these models do not take into account particular consumption pattern of raw materials in batch production, i.e., assume implicitly that the raw materials required for the products are procured in the optimal manner. In practice, the raw materials required cannot be

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determined without knowing the production lot sizes of the products. Hence we know easily the fact that the problem of determining the lot sizes of the products cannot be treated independently of the problem of the procurement policies for the raw materials.

In this regard, Goyal [3] derived an integrated production inventory policy for the case of a single product produced on a single facility. The model simultaneously determines the economic batch size for the product and economic order quantities for raw materials, that will minimize the total variable cost of the production system. As indicated in Korgaonker [5], Goyal's solution takes into account the fact that raw materials are consumed only during the production time of the batch, and not uniformly as commonly assumed. This fact is more important in a case where multiple products are produced on a single (or multiple) facility production system on a batch basis. Korgaonker [5] also studied integrated production inventory model for multiproduct multifacility production system in which instantaneous production is allowed in each facility. This assumption is quite restrictive in practice.

In this paper, an integrated production inventory model is developed for two products on a single facility in which the raw materials are fed to the production system as required at the facility. The objective is to determine simultaneously the optimal production schedule for the products and the corresponding optimal procurement policies for raw materials in a way that minimizes the total variable cost of the production system. The production schedule for the more frequently set-up product is constructed by the method of equal or unequal lot sizes. This method is possible to obtain lower cost solutions.

2. The Mathematical Model

The mathematical model presented is based on the following assumptions:

1. There is only one facility for producing two products, where only one product can be made at any given time.
2. Raw materials are fed to the production system as required at the facility.
3. The raw materials are procured from outside sources and their replenishment rates are assumed to be infinite.
4. The demand rates and production rates for each product are known and constant.
5. Procurement lead-time for the materials is zero, and planning horizon is infinite.
6. Shortage of raw materials or the products is not permitted.
7. All the other cost factors pertaining to set up, inventory holding order processing are assumed known with certainty.

The following notations are used throughout this paper:

For the i-th product (i = 1, 2),

T basic period

V integer number of basic period in common cycle (VT)

p_i production rate per year, assumed constant

d_i demand rate per year, assumed constant ($p_i \geq d_i$)

ρ_i intensity of demand ($= d_i/p_i$)

T_i production cycle time

t_i production time ($= \rho_i T_i$)

H_i inventory holding cost per unit per year

S_i manufacturing set-up cost.

For the j-th raw materials ($j = 1, 2 \ldots n$),

n number of raw materials required for the product

W_j a positive integer such that the raw material is ordered once every W_jVT cycle

h_j stock holding cost per unit per year

s_j cost of placing a procurement

I_j the number of units in material during a cycle (W_jVT)

m_{ij} amount of raw material required to make one unit of the product i ($i = 1,2$).

For the cost function,

$R_j[T,V,W_j]$ annual variable cost for the j-th raw material

$R[T,V]$ total minimum variable cost for all the raw materials

$J[T,V]$ annual variable cost for the products

$F[T,V]$ total variable cost per year ($= J[T,V] + R[T,V]$).

Under the above assumptions and notations, the average cost per year when product i is produced in cycle of length T_i is given by

$$C_i(T_i) = S_i/T_i + (1/2)H_id_i(1-\rho_i)T_i. \tag{1}$$

If each product is treated independently, then the optimal cycle length T_i^* is given by

$$T_i^* = [2S_i / \{H_id_i(1-\rho_i)\}]^{1/2} \tag{2}$$

corresponding to the minimum cost

$$C_i^*(T_i^*) = [2S_iH_id_i(1-\rho_i)]^{1/2}. \tag{3}$$

However, it is not possible to produce each product according to its optimal frequency because of facility interference. It is well known that the sum of independent optimal costs, ΣC_i, is a lower bound on the optimal value of any feasible solution. Consequently, there arises the need to search for a feasible least cost solution.

To obtain feasible schedule for the case of two products, we define the product having smaller cycle of T_i as our first product. That is, let the products be numbered such that $T_1 \leq T_2$. In particular, we will deal directly with the integer number of a basic period (T) as appeared in the previous models [2]. Let V be a positive integer such that $T_1 = T$ and $T_2 = VT$. Then

$$T_2 = VT_1 \tag{4}$$

implies that the first product is produced V times, while the second product is made only once in a total cycle time (VT). If (4) is violated, it is clear that there will be times when both products will simultaneously require time on the facility. The scheme of lot sizes is shown in Figure 1.

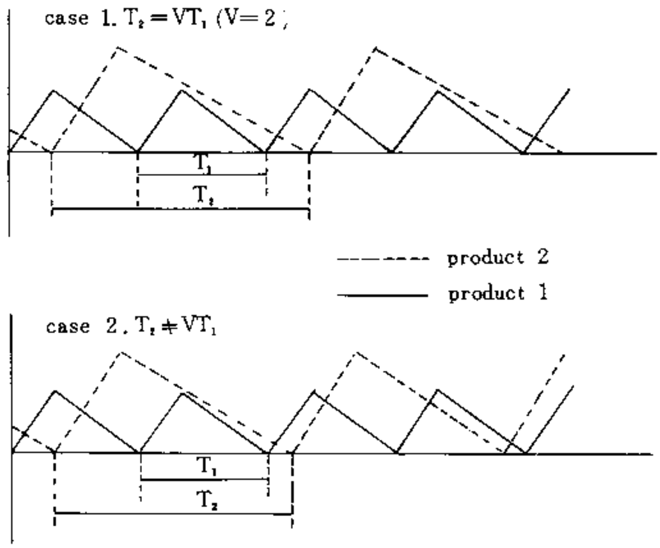

Figure 1. Equal lot size scheme (case 1) and facility interference (case 2)

For schedule feasibility, it is necessary that

$$V \leq X \tag{5}$$

where $X = (1 - \rho_1)/\rho_2$. Note that (5) is derived by $t_1 + t_2 = \rho_1 T + \rho_2 VT \leq T$. We know that conditions (4) and (5) are necessary and sufficient condition for the feasibility of the schedule.

Now, to obtain the optimal policies, we will consider two kinds of cost function in case of the infeasibility ($V > X$) as well as feasibility ($V \leq X$) condition. Note that when $V \leq X$, the production schedule can be generated as the equal lot sizes (Condition 1) for the more frequently manufactured product, and when $V > X$, the production schedule can be modified as the unequal lot sizes (Condition 2).

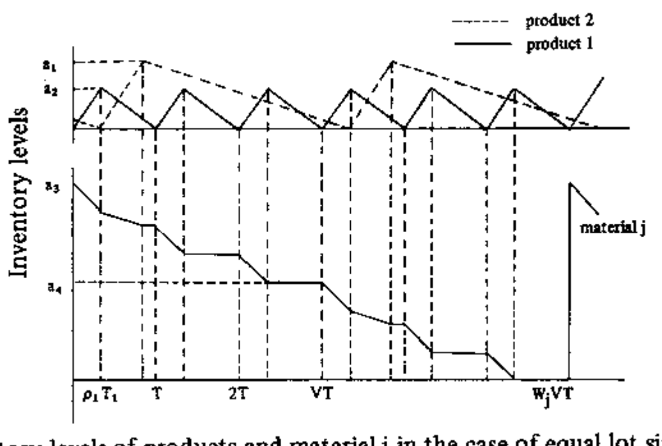

Figure 2. Inventory levels of products and material j in the case of equal lot size (V=3, W_j=2): $a_1 = (1 - \rho_2)d_2 VT$, $a_2 = (1 - \rho_1)d_1 T$, $a_3 = (m_{1j}d_1 + m_{2j}d_2)W_j VT$, $a_4 = (m_{1j}d_1 + m_{2j}d_2)VT$.

2.1. Condition 1

The production system is characterized by undertaking setups at equal time intervals. The one feasible schedule for the equal lot sizes is shown in the Figure 2.

The annual variable cost of the two products from manufacturing set-up and inventory holding is

$$\begin{aligned} J[T,V] &= S_1/T + (1/2)H_1 d_1(1-\rho_1)T + S_2/VT + (1/2)H_2 d_2(1-\rho_2)VT \\ &= (S_1 + S_2/V)/T + (1/2)\{H_1 d_1(1-\rho_1) + H_2 d_2(1-\rho_2)V\}T. \end{aligned} \tag{6}$$

For raw material j, the number of units in inventory during a cycle ($W_j VT$) is

$$I_j = (1/2)W_j VT^2\{d_1 m_{1j}(\rho_1 + VW_j - 1) + d_2 m_{2j}(2\rho_1 + VW_j + \rho_2 V - V)\}. \tag{7}$$

Then the annual variable cost for the j-th raw material $R_j[T,V,W_j]$ from procurement and stock holding becomes

$$R_j[T,V,W_j] = \{s_j + h_j I_j\}/W_j VT. \tag{8}$$

With the given T and V, the corresponding optimal value of $W_j = W_j(T, V)$, defined implicitly, which minimizes $R_j[.]$ is found by

$$R_j[T,V,W_j(T,V)] \leq \min\{R_j[T,V,W_j(T,V)+1],\ R_j[T,V,W_j(T,V)-1]\}. \tag{9}$$

As described in Goyal [3], we assume that $R_j[T, V, W_j]$ can be treated as a continuous function of W_j. By differentiating partially with respect to W_j, we obtain

$$W_j^* = G_j/VT \tag{10}$$

were $G_j = [2s_j/\{h_j(d_1 m_{1j} + d_2 m_{2j})\}]^{1/2}$. Notice that $W_j(T, V)$ can be obtained by evaluating and comparing the annual variable cost for the two integer values W_j nearest to W_j^*. This method will be restated in Section 3.

The total minimum variable cost for all the raw materials is given by

$$\begin{aligned} R[T,V] &= \sum_{j=1}^{n} R_j[T,V,W_j(T,V)] \\ &= \sum_{j=1}^{n} \{s_j + h_j I_j\}/\{W_j(T,V)VT\}. \end{aligned} \tag{11}$$

Hence the average annual total variable cost F[T, V] for the system is given by

$$F[T,V] = J[T,V] + R[T,V]. \tag{12}$$

Differentiating F[T, V] with respect to T and V respectively and equating the first derivative equal to zero, we get

$$T^* = [2\{S_1 + S_2/V + \sum_{j=1}^{n} s_j/VW_j(T,V)\}/A_1]^{1/2} \triangleq T(V) \tag{13}$$

and $$V^* = (1/T)\,[2\{S_2 + \sum_{j=1}^{n} s_j/W_j\}/A_2]^{1/2} \triangleq V(T) \tag{14}$$

where

$$A_1 = H_1 d_1(1-\rho_1) + H_2 d_2(1-\rho_2)V + \sum_{j=1}^{n} h_j[d_1 m_{1j}\{\rho_1 + VW_j(T,V) - 1\} + d_2 m_{2j}\{2\rho_1 + VW_j(T,V) + \rho_2 V - V\}]$$

and

$$A_2 = H_2 d_2(1-\rho_2) + \sum_{j=1}^{n} h_j[d_1 m_{1j} W_j(T,V) + d_2 m_{2j}\{W_j(T,V) + \rho_2 - 1\}].$$

Note that the value of V^* obtained above may not be integer because of the underlying continuity assumption. V^* in (14) can be modified as W_j in (10). In case of a single product, i.e., $d_2 = S_2 = 0$, $V = 1$ and $m_{2j} = 0$ for all j, the results obtained are the same as those found by Goyal [3].

2.2. Condition 2

When the set-up frequency for the first product, V, does not satisfy the condition (5), we know easily the fact that the first product cannot be produced by using the equal time intervals. As in Goyal [4], we will adopt the conception of unequal lot sizes. That is, for the first product one set-up covers a time interval of VT/X and remaining (V-1) set-ups cover a time interval of VT(X-1)/X with equal time intervals. The production system characterized by the unequal lot sizes is represented in the Figure 3.

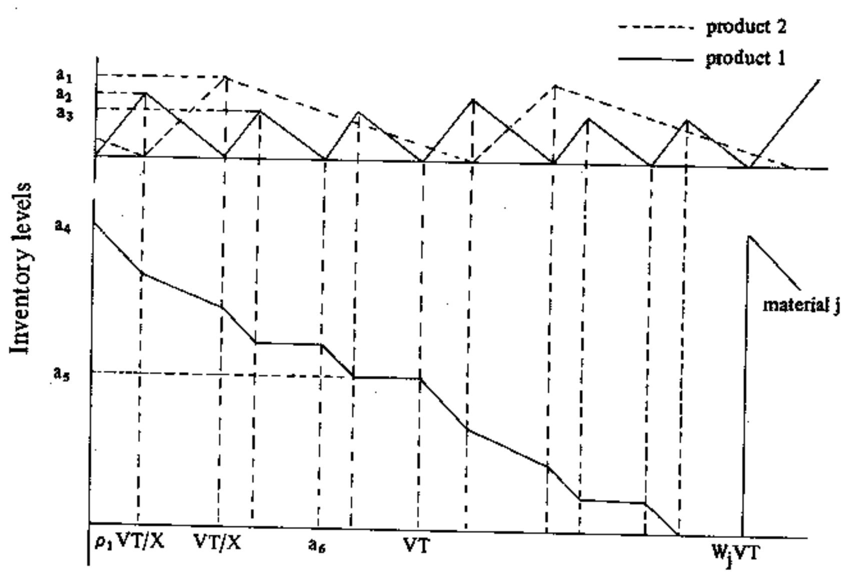

Figure 3. Inventory levels of products and material j in the case of unequal lot size ($V = 3, W_j = 2$): $a_1 = (1-\rho_2)d_2VT$, $a_2 = (1-\rho_1)d_1VT/X$, $a_3 = (1-\rho_1)d_1(X-1)VT/X(V-1)$, $a_4 = (m_{1j}d_1 + m_{2j}d_2)W_jVT$, $a_5 = (m_{1j}d_1 + m_{2j}d_2)VT$, $a_6 = \{1 + (X-1)/(V-1)\}\,VT/X$.

Following procedures similar to those in Condition 1, the annual variable cost of the two products becomes

$$J[T,V] = (S_1 + S_2/V)/T + (1/2)\,[H_1 d_1(1-\rho_1)\{1+(X-1)^2/(V-1)\}/X^2 + H_2 d_2(1-\rho_2)]\,VT \tag{15}$$

and for raw material j,

$$I_j = (1/2)W_j V^2 T^2\,[(d_1 m_{1j}/X^2)\{\rho_1(1+(X-1)^2/(V-1)) + 2(X-1) + (V-2)(X-1)^2/(V-1) + (W_j-1)X^2\} + d_2 m_{2j}\{(\rho_1+1)/X + (W_j-1)\}]. \tag{16}$$

As in Condition 1, the optimal values are given by

$$T^* = [2\{S_1 + S_2/V + \sum_{j=1}^{n} s_j/VW_j(T,V)\}/B_1]^{1/2} \triangleq T(V) \tag{17}$$

and
$$V^* = (1/T)\,[2\{S_2 + \sum_{j=1}^{n} s_j/W_j\}/B_2]^{1/2} \triangleq V(T) \tag{18}$$

where

$$\begin{aligned} B_1 = {} & H_1 d_1 (V/X^2)(1-\rho_1)\{1+(X-1)^2/(V-1)\} + H_2 d_2 V(1-\rho_2) \\ & + \sum_{j=1}^{n} h_j V[(d_1 m_{1j}/X^2)\{\rho_1(1+(X-1)^2/(V-1)) + 2(X-1) \\ & \quad + (V-2)(X-1)^2/(V-1) + X^2(W_j(T,V)-1)\} \\ & \quad + d_2 m_{2j}\{(\rho_1+1)/X + W_j(T,V) - 1\}] \end{aligned}$$

and

$$\begin{aligned} B_2 = {} & H_1 d_1(1-\rho_1)\{1-(X-1)^2/(V-1)^2\}/X^2 + H_2 d_2(1-\rho_2) \\ & + \sum_{j=1}^{n} h_j[(d_1 m_{1j}/X^2)\{\rho_1(1-(X-1)^2/(V-1)^2) + 2(X-1) \\ & \quad + (X-1)^2(1+1/(V-1)^2) + X^2(W_j(T,V)-1)\} \\ & \quad + d_2 m_{2j}\{(\rho_1+1)/X + W_j(T,V)-1)\}]. \end{aligned}$$

3. Solution method

As seen in (9), (13) and (14), we can derive the implicit solution for T, V and W_j. In these formulae, the values of V and W_j are positive integer. However, since the value of V^* and W_j^* obtained above may not be integer, we examine below the question of integrality of V^* and W_j^*.

Since W_j is related by $R_j[.]$ in (8), we assume that $R_j[W_j]$ be the function of W_j. Let W_j be the largest integer less than or equal to W_j^* in (10). We wish to derive conditions for rounding off W_j^* to either W_j or $W_j + 1$, so that we can write $W_j^* = W_j + e_j$ $(0 \leq e_j < 1)$. Then we also know easily that W_j^* is rounded off to W_j if $R_j[W_j] < R_j[W_j + 1]$. That is,

$$W_j^* = \begin{cases} W_j & \text{if } R_j[W_j] < R_j[W_j+1] \\ W_j + 1 & \text{if } R_j[W_j] > R_j[W_j+1] \\ W_j \text{ or } W_j + 1 & \text{if } R_j[W_j] = R_j[W_j+1]. \end{cases} \quad (19)$$

The integrality of V* can be treated in an identical manner.

$$V^* = \begin{cases} V & \text{if } F[V] < F[V+1] \\ V + 1 & \text{if } F[V] > F[V+1] \\ V \text{ or } V + 1 & \text{if } F[V] = F[V+1]. \end{cases} \quad (20)$$

Now, the steps of the computational scheme are proposed below.

Stage A: <For Condition 1 : Equal Lot Sizes>

- Step A1. (Initialization) For all j, set $W_j = 1$.
  Set $V = 1$.
- Step A2. Find T* in (13).
- Step A3. Compute V* by (14) and round off V*.
- Step A4. For all j, compute W_j^* by (10) and round off W_j^*.
- Step A5. Repeat Step A2-A4 until convergence is obtained.
- Step A6. Go to Stage B.

Stage B: <Checking of feasible schedule>

- Step B1. If $V \leq X$, then Stop (feasible solution).
  Otherwise, go to Step B2.
- Step B2. (Initialization) Set $V = [X]$, where [.] = Gauss value.
  For all j, Set $W_j = 1$.
- Step B3. Find T* from (13).
- Step B4. For all j, compute W_j^* by (10) and round off W_j^*.
- Step B5. Repeat Step B3-B4 until convergence is obtained.
- Step B6. Compute F[T, V] from (12). Set TC1 = F[T, V].

Stage C: <For Condition 2 : Unequal Lot Sizes>

- Step C1. (Initialization) For all j, set $W_j = 1$.
  Set $V = [X] + 1$.
- Step C2. Find T* from (17).
- Step C3. Compute V* by (18) and round off V*.
- Step C4. For all j, compute W_j^* by (10) and round off W_j^*.
- Step C5. Repeat Step C2-C4 until convergence is obtained.
- Step C6. Compute F[T, V]. Set TC2 = F[T, V].
  If TC1 > TC2, then the solution becomes unequal lot sizes. Otherwise, the solution become equal lot sizes in Stage B.

4. Numerical Example

To illustrate the computational scheme developed, a numerical example is considered. This procedure is coded in Basic and run on an IBM-AT personal computer. The input data used for this example is shown in Table 1. In this example, we assume that the raw materials (1,2,3) can be used to make product 1, and the product 2 can be manufactured within the raw materials (2,3,4).

Table 1. Input data for the example

product i	p_i	d_i	S_i	H_i
1	200,000	100,000	27,000	20
2	180,000	60,000	42,000	8
material j	m_{1j}	m_{2j}	s_j	h_j
1	1	0	13,000	0.6
2	4	6	9,000	0.1
3	5	3	8,000	0.2
4	0	2	14,500	0.3

The values of V's and W_j's obtained at different iterations are given in Table 2. In applying the Stage A, we first set all W_j and V equal to unity, and determine production cycle time T* = 0.38097 years. Using this value, V* is recomputed. Next, all W_j* are recomputed. Substitution of the latter into expression for T*, yields new value of T*, i.e., T* = 0.25744 years. The values of V* and W_j* converge at third iteration. Then the optimal values are given by T* = 0.24655 year, V* = 2, $(W_1, W_2, W_3, W_4)^* = (1,1,1,2)$. But, since V* > X(= 1.5), we can check easily that the production schdule is not feasible. Therefore we must consider the next stages. Following stages similar to those in Stage A.

It can be seen from the results that the minimum cost occurs in Stage B when T* = 0.32369 year, V* = 1, $(W_1, W_2, W_3, W_4)^* = (2,2,1,3)$. The corresponding optimal total variable cost is \$146784 per year.

Table 2. Results of computational scheme

Stage A:

Iteration	V	W_1	W_2	W_3	W_4	T
0	1	1	1	1	1	0.38097
1	2	1	1	1	1	0.25744
2	2	1	1	1	2	0.24655
3	2	1	1	1	2	0.24655*

The corresponding total cost = \$143590 per year.

Stage B:

Iteration	V	W_1	W_2	W_3	W_4	T
0	1	1	1	1	1	0.38097
1	1	2	1	1	2	0.34667
2	1	2	1	2	3	0.33879
3	1	2	2	1	3	0.32369
4	1	2	2	1	3	0.32369*

The corresponding total cost = $146784 per year**.

Stage C:

Iteration	V	W_1	W_2	W_3	W_4	T
0	2	1	1	1	1	0.24956
1	2	1	1	1	2	0.23924
2	2	1	1	1	2	0.23924*

The corresponding total cost = $146858 per year.

Note: * = convergence, ** = minimum cost.

5. Conclusion

An integrated production inventory model has been developed in case of two ptoducts on a single facility. The model simultaneously determined the optimal production schedule for the products and the corresponding optimal procurement policies for the raw materials. In particular, production schedule was generated by the equal or unequal lot sizes.

In addition, a computational procedure was developed for finding the optimal solution. To illustrate the procedure, a simple example was solved.

More work could be done to extend the result of this paper. For example, multiproduct situation and decaying raw materials could be considered. In a subsequent paper, we intend to study the model of grouping the raw materials. This problem becomes desirable for multiple raw material inventory management due to several factors such as savings on order placing costs, better implementation of order control, and the availability of group discounts.

References

[1] Doll, C. L. and Whybark, D. C., "An Interative Procedure for the Single-Machine Multi-Product Lot Scheduling Problem," *Management Science*, Vol. 20, No. 1, pp. 50-55, 1973.

[2] Elmaghraby, S. E., "The Economic Lot Scheduling Problem (ELSP) : Review and Extention," *Management Science*, Vol. 24, No. 6, pp. 587-598, 1978.

[3] Goyal, S. K., "An Integrated Inventory Model for a Single Production System," *Operational Rese-*

arch Quarterly, Vol. 28, No. 3, pp. 539-545, 1977.

[4] Goyal, S. K., "Determination of Economic Production Quantities for a Two-Product Single Machine System," *International Journal of Production Research*, Vol. 22, No. 1, pp. 121-126, 1984.

[5] Korgaonker, M. G., "Integrated Production Inventory Policies for Multistage Multiproduct Batch Production Systems," *Journal of the Operational Research Society*, Vol. 30, No. 3, pp. 355-362, 1979.

[6] Seipe, A. R., "Production Runs for Multiple Products: The two-Product Heuristic," *Management Science*, Vol. 23, No. 12, pp. 1321-1327, 1977.

[7] Vemuganti, R. R., "On the Feasibility of Scheduling Lot Sizes for Two Products on One Machine," *Management Science*, Vol. 24, No. 15, pp. 1668-1673, 1978.