Journal of the KSQC
Vol. 14, No. 2, 1986

An Asymptotically Efficient Test for Exponential Populations**

Jong Woo Jeon*
Han Young Chung*
Youn Tae Kim*

ABSTRACT

Using Fisher's method of combining two independent test statistics, we suggest a test for comparing two exponetial populations with location and scale parameters and prove that it is asymptotically optimal in the sense of Bahadur efficiency.

I. INTRODUCTION

For the usual life testing situation in which observations become available in an ordered manner, it is natural to consider the possibility of terminating the life test at an early stage; say, after the first r out of n observations have been recorded. Any such procedure has the advantage that it may lead to a decision in a shorter period of time and with fewer observations than a procedure which involves observing what happens to all the items being tested.

For a particular life testing problem involving the two parameter exponential distribution, Perng (1977) has proposed an asymptotically optimal procedure based on the first r out of n failure time. In this paper, we show that the procedure used in Perng (1977) can also be applied to the corresponding two exponential populations problem.

Let two ordered samples $X_1, \ldots, X_r$, and $Y_1, \ldots, Y_l$ be taken from exponential populations

$$f(x, \beta_1, \sigma_1) = \begin{cases} \dfrac{\exp\{-(x-\beta_1)/\sigma_1\}}{\sigma_1}, & x \geq \beta_1 \\ 0, & \text{otherwise} \end{cases}$$

and

* Department of Computer Science and Statistics, Seoul National University.
** This research was supported by the MOE, through the Research Institute for Basic Sciences, SNU 1985-1986.

$$f(y, \beta_2, \sigma_2)= \begin{cases} \dfrac{\exp\{-(y-\beta_2)/\sigma_2\}}{\sigma_2} & , y \geq \beta_2 \\ 0 & , \text{otherwise} \end{cases}$$

respectively. Let $\theta=(\beta_1, \beta_2, \sigma_1, \sigma_2)$ and let

$$\Omega=\{\theta: 0 \leq \beta_{i0} \leq \beta_i < \infty,\ 0 < \sigma_i^2 < \infty,\ i=1,2\}.$$

The statistical problem is to test

$$H_0: \beta_1+\beta_2=0,\ \sigma_1=\sigma_2$$

against

$$H_1: \beta_1+\beta_2 > 0,\ \sigma_1 \neq \sigma_2$$

on the basis of the first r, l order statistics.

There are cases of practical interest to test the scale parameter and the location parameter at the same time. For example, suppose that two electronic components are to be used. The company is interested in determining whether or not the sum of the guaranted lifetime of two electronic components is above a given criterion $(\beta_1+\beta_2 > \beta_{10}+\beta_{20})$ and variations of the lifetime of two electronic components are not equal $(\sigma_1 \neq \sigma_2)$. It is assumed that the lifetime of electronic components can be well approximated by an exponential distribution.

This paper suggests a test for the problem mentioned above, based on $X_1, \cdots, X_r$ and $Y_1, \cdots, Y_l$, and we prove that the suggested test is asymptotically optimal in the sense of Bahadur efficiency.

II. THE TEST

First we define the following notation:

$$U_{r,l} = \frac{mX_1+nY_1}{\sum_{i=1}^{r} X_i+(m-r)X_r-mX_1+\sum_{i=1}^{l} Y_i+(n-l)Y-nY_1}$$

$$F_{r,l} = \frac{\sum_{i=1}^{r} X_i+(m-r)X_r-mX_1/(r-1)}{\sum_{i=1}^{l} Y_i+(m-l)Y_l-nY_1/(l-1)}$$

$$H_{r,l} = \begin{cases} 2\{1-G_{r,l}(F_{r,l})\}, & \text{if } F_{r,l} \geq \text{med}\, G_{r,l} \\ 2G_{r,l}(F_{r,l}), & \text{if } F_{r,l} \geq \text{med}\, G_{r,l} \end{cases}$$

$$W_{r,l} = -2\log H_{r,l}$$

where $G_{r,l}$ is the distribution function of central F with $2r$-2 and $2l$-2 degrees of freedom, and med $G_{r,l}$ is the median of central F with $2r$-2 and $2l$-2 degrees of freedom. We note that $U_{r,l}$ is *a* modified likelihood ratio test statistics for testing $H_0 : \beta_1=\beta_2=0$ against $H_1 : \beta_1+\beta_2\neq 0$ when $\sigma_1=\sigma_2=\sigma$ is unknown and $F_{r,l}$ is the likelihood ratio test statistics for testing $H_0 : \sigma_1=\sigma_2$ against $H_1 : \sigma_1\neq\sigma_2$ when β_1 and β_2 are unknown.

The following two lemmas are important and their proofs are similar to those in Perng (1977).

Lemma 2.1: The statistics $U_{r,l}$ and $W_{r,l}$ are independent if $\sigma_1=\sigma_2$

Lemma 2.2: Under H_0, $F_{r,l}$ has the central F distribution with $2r$-2 and $2l$-2 degrees of freedom and $U_{r,l}$ has the following probability density function;

$$g(z) = \begin{cases} \frac{\Gamma(r+l)}{\Gamma(r+l-2)} z\left(\frac{1}{1+z}\right)^{r+l}, & z \geqq 0 \\ 0 & , z < 0 \end{cases} \tag{2.1}$$

The following lemma is due to Perng and Littell (1976).

Lemma 2.3: Under H_0, $W_{r,l}$ has the central chi-squared distribution with 2 degrees of freedom.

Now $U_{r,l}$ is used to test $H_0 : \beta_1+\beta_2=0$ against $H_1 : \beta_1+\beta_2>0$, two-sided F test is seems reasonable to combine the two test statistics by Fisher's method (Fisher, 1950, pp. 99-101) as follows:

Reject H_0 if $Q_{r,l}\geq C_0$

Accept H_0 if $Q_{r,l} < C_0$

where $Q_{r,l}= -2\log P_{H0}(U_{r,l}>u)-2\log P_{H0}(W_{r,l}>w)$, u and w are the observed values of $U_{r,l}$ and $W_{r,l}$ respectively. It is known that under H_0, $Q_{r,l}$ has a chi-squared distribution with 4 degrees of freedom. Thus, the critical value C_0 can be easily determined by using a chi-square table.

III. AN ASYMPTOTICALLY OPTIMAL PROPERTY OF TEST

In this section, we show that the test given in section 2 is asymptotically optimal in the sense of Bahadur efficiency.

Throughout this section, we shall assume that $m/N\rightarrow\lambda$, $r/m\rightarrow\lambda_1$, and $l/n\rightarrow\lambda_2$ as $N\rightarrow\infty$ where $N=m+n$, $0<\lambda, \lambda_1, \lambda_2\leq 1$. The following lemmas give the exact slope of $\{U_{r,l}\}$, $\{W_{r,l}\}$ and $\{Q_{r,l}\}$

Lemma 3.1: Under H_1, the exact slope of $\{U_{r,l}\}$ and $\{W_{r,l}\}$ are

$$c_1(\theta) = 2\log\left(\frac{\lambda\lambda_1\sigma_1+(1-\lambda)\lambda_2\sigma_2+\lambda\beta_1+(1-\lambda)\beta_2}{\lambda\lambda_1\sigma_1+(1-\lambda)\lambda_2\sigma_2}\right)^{\{\lambda\lambda_1+(1-\lambda)\lambda_2\}} \tag{3.2}$$

and

$$c_2(\theta)=2\log\left(\frac{\{\lambda\lambda_1\sigma_1+(1-\lambda)\lambda_2\sigma_2\}^{\lambda\lambda_1+(1-\lambda)\lambda_2}}{(\lambda\lambda_1+(1-\lambda)\lambda_2)^{(\lambda\lambda_1+(1-\lambda)\lambda_2)}\sigma_1^{\lambda\lambda_1}\sigma_2^{(1-\lambda)\lambda_2}}\right) \tag{3.3}$$

respectively.

(Proof): The exact slope $c_1(\theta)$ of $\{U_{r,l}\}$ will be obtained by using Theorem 7.2 of Bahadur (1971). Let $U^*_{r,l}=\sqrt{N}\,U_{r,l}$. Then,

$$\lim_{N\to\infty}(1/N)\,U^*_{r,l}=\frac{\lambda\beta_1+(-\lambda)\beta_2}{\lambda\lambda_1\sigma_1+(1-\lambda)\lambda_2\sigma_2} \tag{3.4}$$

with probability one under H_1. Furthermore

$$\lim_{N\to\infty}(1/N)\log P_{H0}(U^*_{r,l}\geq Nt)$$

$$=\lim_{N\to\infty}(1/N)\log P_{H0}(U_{r,l}\geq t) \tag{3.5}$$

Since $\log P_{H0}(U_{r,l}\geq t)$

$$=\log[(1+t)^{-(r+l)+1}\{1+(r+l-1)t\}],$$

(3.5) becomes

$$\lim_{N\to\infty}(1/N)\log P_{H0}(U^*_{r,l}\geq\sqrt{N}t)$$

$$=\lim_{N\to\infty}(1/N)\log(1+t)^{-(r+l)+1}+\lim_{N\to\infty}(1/N)\log\{1+(r+l-1)t\}$$

$$=-\{\lambda\lambda_1+(1-\lambda)\lambda_2\}\log(1+t) \tag{3.6}$$

Using (3.4) and (3.6), the exact slope $c_1^*(\theta)$ of $\{U^*_{r,l}\}$ can be obtained by Theorem 7.2 of Bahadur (1971) as

$$c_1^*(\theta)=2\log\left\{\frac{\lambda\lambda_1\sigma_1+(1-\lambda)\lambda_2\sigma_2+\lambda\beta_1+(1-\lambda)\beta_2}{\lambda\lambda_1\sigma_1+(1-\lambda)\lambda_2\sigma_2}\right\}^{\lambda\lambda_1+(1-\lambda)\lambda_2} \tag{3.7}$$

Since $U^*_{r,l}$ is a strictly increasing function of $U_{r,l}$ $\{U_{r,l}\}$ and $\{U^*_{r,l}\}$ have the same exact slope (eg. See Bahadur, 1971, p. 27). Hence, $c_1(\theta)$ becomes (3.7).

The exact slope of $\{W_{r,l}\}$ is computed as follows: Clearly under H_1,

$$c_2(\theta)=\lim_{N\to\infty}(-1/N)\log H_{r,l}$$

with probability one. Now if $\sigma_1>\sigma_2$ then with probability one, $H_{r,l}=2\{1-G_{r,l}(F_{r,l})\}$ for sufficiently large N. From Killen, Hettmansperger, and Sievers (1972), for $t>1$

$$\lim_{N\to\infty}(1/N)\log\{1-G_{r,t}(t)\}$$

$$=\log\left[\left\{\frac{\lambda\lambda_1+(1-\lambda)\lambda_2}{(1-\lambda)\lambda_2+\lambda\lambda_1 t}\right\}^{\lambda\lambda_1+(1+\lambda)\lambda_2} t^{\lambda\lambda_2}\right]$$

and $F_{r,t}$ tends to σ_1/σ_2 with probability one. Thus if $\sigma_1 > \sigma_2$, we have

$$c_2(\theta)=\lim_{N\to\infty}(-2/N)\log\{1-G_{r,t}(F_{r,t})\}$$

$$=2\log\left(\frac{\lambda\lambda_1+(1-\lambda)\lambda_2\sigma_2^{\lambda\lambda_1+(1-\lambda)\lambda_2}}{(\lambda\lambda_1+(1+\lambda)\lambda_2)^{\lambda\lambda_1+(1-\lambda)\lambda_2}\sigma_1^{\lambda\lambda_1}\sigma_2^{(1-\lambda)\lambda_2}}\right) \qquad (3.8)$$

Similarly for $\sigma_1 < \sigma_2$, $c_2(\theta)$ becomes (3.8).

The following lemma gives the exact slope of $\{Q_{r,t}\}$.

Lemma 3.2: Under H_1, the exact slope $\{Q_{r,t}\}$ is

$$c_Q(\theta)=2\log\left(\frac{\{\lambda\lambda_1\sigma_1+(1-\lambda)\lambda_2\sigma_2+\lambda\beta_1+(1-\lambda)\beta_2\}^{(\lambda\lambda_1+(1-\lambda)\lambda_2)}}{(\lambda\lambda_1+(1-\lambda)\lambda_2^{(\lambda\lambda_1+(1-\lambda)\lambda_2)}\sigma_1^{\lambda\lambda_1}\sigma_2^{(1-\lambda)\lambda_2}}\right) \qquad (3.9)$$

(Proof): Equation (3.9) follows from results of Little and Folks (1971) which show that

$$c_Q(\theta)=c_1(\theta)+c_2(\theta), \quad \text{under} \quad H_1.$$

Before we state the main theorem, let Ω_i, $i=0,1$ be the set parameters corresponding hypothesis H_i, $i=0,1$ respectively and let $f(X_r^*, Y_t^*, \theta_i)$ be the joint density of $(X_1,\cdots X_r)$ and $(Y_1,\cdots Y_t)$ when $\theta_i\in\Omega_i$, $i=0,1$.

Theorem: For the hypotheses testing problem defined in Section 1, the Q test proposed in Section 2 is asymptotically optimal in the sense of Behadure efficiency.

(Proof): For every $\theta_i\in\Omega_i$, $i=0,1$, define

$$h(X_r^*, Y_t^*\ \theta_1, \theta_0)=\frac{f(X_r^*, Y_t^*, \theta_1)}{f(X_r^*, Y_t^*\ \theta_0)}$$

Let the quantity $K^{(1)}(\theta_1, \theta_0)$ be given by

$$K^{(1)}(\theta_1,\theta_0)=\lim_{N\to\infty}(1/N)\log h(X_r^*, Y_t^*, \theta_1, \theta_0)$$

Then,

$$K^{(1)}(\theta_1,\theta_0)$$

$$=\lambda\lambda_1\log(1/\sigma_1)+(1-\lambda)\lambda_2\log(1/\sigma_2)-\lambda\lambda_1-(1-\lambda)\lambda_2$$

$$-(\lambda\lambda_1+(1-\lambda)\lambda_2)\log(1/\sigma)+(1/\sigma)\{\lambda\lambda_1\sigma_1+(1-\lambda)\lambda_2\sigma_2$$

$$+\lambda\beta_1+(1-\lambda)\beta_2\},$$

Define

$$J(\theta_1)=\inf\{K(\theta_1,\theta_0)\mid\theta_0\in\Omega_0\}.$$

Then,

$$J(\theta_1)=\log\left(\frac{\{\lambda\lambda_1\sigma_1+(1-\lambda)\lambda_2\sigma_2+\lambda\beta_1+(1-\lambda)\beta_2\}^{(\lambda\lambda_1+(1-\lambda)\lambda_2)}}{(\lambda\lambda_1+(1-\lambda)\lambda_2)^{(\lambda\lambda_1+(1-\lambda)\lambda_2)}\sigma_1^{\lambda\lambda_1}\sigma_2^{(1-\lambda)\lambda_2}}\right)$$

Thus we have

$$c_Q(\theta_1)=2J(\theta_1),$$

By Corollary 3 of Bahadur and Raghavachari (1972), this proves the theorem.

REFERENCES

1. Bahadur, R.R. (1971). Some Limit Theorems in Statistics. Philadelphia: SIAM.
2. Bahadur, R.R. and Raghvachari, M. (1972). Some asymptotic properties of likelihood ratios on general sample spaces. Proc. Sixth Berkeley Symp. Math. Statist. Prob. Berkeley and Los Angeles: Univ. of California press, 129-52.
3. Fisher, R.A. (1950). Statistical Methods for Research Workers. (11th ed.), London: Oliver and Boyed.
4. Killen, T.J., Hettmansperger, T.P. and Sievers, G.L. (1972). An elementary theorem on the probability of large deviations. Ann. Math. Statist. 43, 181-92.
5. Littell, R.C. and Folks, J.L. (1971). Asymptotic optimality of Fisher's method of combining independent tests. J. Amer. Statist. Assoc., 66, 802-6.
6. Littell, R.C. and Folks, J.L. (1973). Asymptotic optimality of Fisher's method of combining independent test. J. Amer. Statist. Assoc., 68, 193-4.
7. Perng, S.K. (1977). An asymptotically efficient test for the location parameter and the scale parameter of an exponential distribution. Commun. Statist. Theor. Meth., A6 (14), 1399-407.
8. Perng, S.K. and Littell, R.C. (1976). A test of equality of two normal population means and variances. J. Amer. Statist. Assoc., 71, 968-70.