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ON A CLASS OF MINIMAL HP-SPACES

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1. Introduction

A *P-space* is a topological space in which every G_δ-set is open. A *HP-space* is a Hausdorff topological space which is also a P-space. A HP-space (X,τ) is said to be *minimal HP* if and only if $\tau' \leqq \tau$ and (X,τ') is a HP-space implies $\tau=\tau'$. Our attention was drawn to P-spaces when we noticed that *a Lindelöf HP-space is maximal Lindelöf and minimal HP.* This result is given by Cameron [1, Theorem 7.6] and Misra [3, Proposition 4.2 (f)]. A closely related class of spaces is the class of HP-closed spaces. A HP-space is called *HP-closed* if and only if it is closed in every HP-space into which it can be embedded. In an earlier paper [4] we made extensive investigations of minimal HP and HP-closed spaces. Cameron has shown that a maximal Lindelöf space need not be Hausdorff [1, Example 7.3] but is a P-space [1, Theorem 7.4] that a minimal HP-space need not be Lindelöf [1, Example 9.1] and that a Lindelöf HP-space need not be minimal Hausdorff nor minimal P. We have given an example [4, Example 2.2] to show that the classes of HP-closed and H-closed spaces are distinct.

If $\mathscr{C}$ is a family of subsets of a topological space (X,τ), then $\mathscr{C}$ ic called an *almost cover* of X if the collection of closures of members of $\mathscr{C}$ covers X. $\mathscr{C}$ is said to have an *almost subcover* $\mathscr{C}'$ of X if $\mathscr{C}'\subset\mathscr{C}$ and $\mathscr{C}'$ is an almost cover in its own right. We say that a filter (base) is a *σ-filter* (*base*) if the filter (generated by the filter base) is closed under countable intersections. *A HP-space (X,τ) is HP-closed if and only if every open σ-filter base on (X,τ) has an adherent point* (*or equivalently, every open cover has a countable almost subcover*). A subset U of a topological space X is called *regular-open* if and only if int cl $U=U$; a subset is *regular-closed* if and only if its complement is regular-open. A topological space (X,τ) is called *semi-regular* if and only if the family of all regular-open subsets of X forms a base for the topology τ. Indeed *a HP-space*

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(X,τ) is minimal HP if and only if every open σ-filter base with unique point of adherence converges to it (or equivalently, (X,τ) is HP-closed and semi-regular).

In this paper a class of minimal *HP*-spaces is introduced. Such spaces are called α-minimal *HP*-spaces. As the name suggests, these spaces are minimal *HP*; furthermore any Lindelöf *HP*-space is α-minimal *HP*. But there are minimal *HP*-spaces which are not α-minimal *HP* and α-minimal *HP*-spaces which are not Lindelöf. These α-minimal *HP*-spaces are characterized by the elegant property that each of their T_1 p-quotients is μ-Lindelöf (see Definition 2.2 (a) below). Furthermore we show that X is α-minimal *HP* if and only if X is *HP*-closed and rim-α (see Definition 2.2 (b) below).

2. α-Minimal *HP* spaces

DEFINITION 2.1 Let (X,τ) be a *HP*-space. (X,τ) is called *α-minimal HP-space* if and only if given a closed subset A of X and an open cover $\mathscr{C}$ of A, $\mathscr{C}$ has a countable almost subcover $\mathscr{C}'$ of A (i.e. there exists a countable subfamily $\mathscr{C}'\subset\mathscr{C}$ such that $A\subset\bigcup\{\text{cl}\, U \mid U\in\mathscr{C}'\}$).

DEFINITIONS 2.2. (a) Let us call a topological space *μ-Lindelöf* if it is a *P*-space and every open σ-filter base with unique adherent point converges to its adherence. (It is to be noted that μ-Lindelöf spaces need not be Hausdorff; but μ-Lindelöf Hausdorff spaces are precisely minimal *HP*-spaces).

(b) A *HP*-space is defined to be *rim-α* if and only if there exists a nbd system $\{V\}$ of open sets for each point of X with the property that given a closed subset $Q\subset\text{cl}\, V-V$ and an open cover $\mathscr{U}$ (open in X) of Q there exists a countable subfamily $\mathscr{U}'(\subset\mathscr{U})$ such that $Q\subset\bigcup\{\text{cl}\, U \mid U\in\mathscr{U}'\}$. (nbd stands for neighbourhood).

The notion of semi-normal spaces was introduced by Viglino [6]. A topological space X is said to be *semi-normal* if and only if given a closed subset $C\subset X$ and an open set G containing C there exists a regular-open set U with $C\subset U\subset G$ (or equivalently, every closed set has a base consisting of regular-open sets). A semi-normal space need neither be normal nor regular; they are not even closed hereditary; they are not productive.

The concept of r-accumulation points of filters was introduced by Herrington and Long [2]. With particular reference to σ-filter bases we say that a σ-filter base $\mathscr{F}$ in A ($\subset X$, where X is a topological space) *r-accumulates* to $a\in A$ if and only if for each open nbd U of a (open in X) and for each $F\in\mathscr{F}$, $F\cap\text{cl}$

$U \neq \phi$. (It should be noted that "$\mathscr{F}$ in A" means that the elements of $\mathscr{F}$ are nonempty subsets of A). If we denote the adherence of a σ-filter base $\mathscr{F}$ by adh$(\mathscr{F})$ and $x \in$ adh$(\mathscr{F})$, then x is an r-accumulation point of $\mathscr{F}$. Clearly if adh$(\mathscr{F}) \neq \phi$, then the r-accumulant of $\mathscr{F}$ (=the set of all r-accumulation points of $\mathscr{F}$) is nonempty. Let us denote the r-accumulant of $\mathscr{F}$ by $r-\mathrm{acc}(\mathscr{F})$. A σ-filter base $\mathscr{F}$ is said to be *adherent convergent* if and only if U is open and $U \supset \mathrm{adh}(\mathscr{F})$ implies $U \supset F \supset \mathrm{adh}(\mathscr{F})$ for some $F \in \mathscr{F}$. Similarly, a σ-filter base $\mathscr{F}$ is said to be *r-accumulant convergent* if and only if U is open and $U \supset r-\mathrm{acc}(\mathscr{F})$ implies $U \supset F \supset r-\mathrm{acc}(\mathscr{F})$ for some $F \in \mathscr{F}$.

It is clear from the definition of α-minimal *HP*-spaces that *if* (X, τ) *is* α-*minimal HP, then* (X, τ) *is HP-closed*. Let us now prove the following proposition.

PROPOSITION 2.3. *Let* (X, τ) *be a HP-space. Then the following are equivalent*:

(*i*) *X is HP-closed*

(*ii*) *Every* σ-*filter base* $\mathscr{F}$ *in X has a r-accumulation point* (*i.e.* $r-acc(\mathscr{F}) \neq \phi$).

PROOF. (i)$\Rightarrow$(ii). Let $\mathscr{F}$ be a σ-filter base such that $r-\mathrm{acc}(\mathscr{F})=\phi$. Then if $x \in X$, there exists an open nbd U_x of x and $F_x \in \mathscr{F}$ such that $(\mathrm{cl}\, U_x) \cap F_x = \phi$. Thus $\mathscr{C} = \{U_x \mid x \in X\}$ is an open cover of X. Consider $\{F_x \mid x \in X\} \subset \mathscr{F}$. Let B be an arbitrary countable subset of X. Then there exists $F_B \in \mathscr{F}$ such that $F_B \subset \cap \{F_x \mid x \in B\}$. Clearly, $F_B \cap (\cup \{\mathrm{cl}\, U_x \mid x \in B\}) = \phi$ for any countable subset B of X. Thus $\mathscr{C}$ does not admit a countable almost subcover of X. Hence X is not *HP*-closed.

(ii)$\Rightarrow$(i). If X is not *HP*-closed, then there exists an open cover $\mathscr{C}$ of X such that for every countable subset $\mathscr{C}'(\subset \mathscr{C})$, $P(\mathscr{C}') = X - \cup \{\mathrm{cl}\, U \mid U \in \mathscr{C}'\} \neq \phi$. Thus $\beta = \{P(\mathscr{C}') \mid \mathscr{C}'$ is a countable subset of $\mathscr{C}\}$ is a σ-filter base. Further $\{X - \mathrm{cl}\, U \mid U \in \mathscr{C}\} \subset \beta$. Let $x \in X$. There exists $U_x \in \mathscr{C}$ such that $x \in U_x$. Thus, since $(X - \mathrm{cl}\, U_x) \cap \mathrm{cl}\, U_x = \phi$, x is not an r-accumulation point of β. Since the choice of $x \in X$ is arbitrary, $r-\mathrm{acc}(\beta) = \phi$.

THEOREM 2.4. *Let* (X, τ) *be a HP-space. Then the following are equivalent*:

(*i*) *X is* α-*minimal HP*.

(*ii*) *Every* σ-*filter base consisting of open or regular-open sets converges to its adherence*.

(*iii*) *If A is a closed subset of X and* $\mathscr{G}$ *is a* σ-*filter base consisting of open or* *regular-open subsets of X with nonempty traces on A, then $\mathscr{G}$ has a point of adherence in A (i.e. $adh(\mathscr{G})\cap A\neq\phi$).*

(iv) Given any open cover $\mathscr{C}$ of X and $U\in\mathscr{C}$ there exists a countable subfamily $\mathscr{C}'\subset\mathscr{C}-\{U\}$ with $X=U\cup(\cup\{cl\ V\mid V\in\mathscr{C}'\})$.

PROOF. (i)⇒(ii). Let $\mathscr{F}$ be a σ-filter base consisting of open or regular-open subsets of X. Let $\text{adh}(\mathscr{F})=\cap\{\text{cl}\ U\mid U\in\mathscr{F}\}=C$. Let V be an open subset of X such that $V\supset C$. Then $\{X-\text{cl}\ U\mid U\in\mathscr{F}\}$ forms an open cover of $X-C$ and hence of $X-V$. Then there exists a countable subset $\mathscr{B}\subset\mathscr{F}$ such that $\cup\{\text{cl}(X-\text{cl}\ U)\mid U\in\mathscr{B}\}\supset X-V$. Thus $V\supset\cap\{\text{int cl}\ U\mid U\in\mathscr{B}\}\supset\cap\{U\mid U\in\mathscr{B}\}\supset W$ for some $W\in\mathscr{F}$, since $\mathscr{F}$ is a σ-filter base.

(ii)⇒(iii). If $\text{adh}(\mathscr{G})\cap A=\phi$, then $X-A$ is an open set containing $\text{adh}(\mathscr{G})$. Hence by (ii) there exists $U\in\mathscr{G}$ such that $X-A\supset U$ so that $U\cap A=\phi$, a contradiction.

(iii)⇒(i). Let A be a closed subset of X and $\mathscr{C}$ an open cover of A such that there exists no countable subset $\mathscr{C}'\subset\mathscr{C}$ such that $A\subset\cup\{\text{cl}\ U\mid U\in\mathscr{C}'\}$. Thus if $X-\cup\{\text{cl}\ U\mid U\in\mathscr{C}'\}=F(\mathscr{C}')$, $\{F(\mathscr{C}')\mid\mathscr{C}'$ is a countable subset of $\mathscr{C}\}$ is an open σ-filter base $\mathscr{G}$ such that each member of $\mathscr{G}$ has nonempty traces with A. By hypothesis, $\text{adh}(\mathscr{G})\cap A\neq\phi$, so that $(\cap\{\text{cl}(X-\text{cl}\ U)\mid U\in\mathscr{C}\})\cap A\neq\phi$. Hence $[X-\cup\{\text{int cl}\ U\mid U\in\mathscr{C}\}]\cap A\neq\phi$. That is $A\not\subset\cup\{\text{int cl}\ U\mid U\in\mathscr{C}\}$. Hence $\mathscr{C}$ is not an open cover of A, a contradiction.

(i)⇒(iv). Let X be α-minimal *HP*. Let $\mathscr{C}$ be an open cover of X and $U\in\mathscr{C}$. Clearly $X-U$ is closed and $\mathscr{C}-\{U\}$ is an open cover of $X-U$. Hence there exists a countable subfamily $\mathscr{C}'\subset\mathscr{C}-\{U\}$ so that $X-U\subset\cup\{\text{cl}\ V\mid V\in\mathscr{C}'\}$. Thus $X=U\cup(\cup\{\text{cl}\ V\mid V\in\mathscr{C}'\})$.

(iv)⇒(i). Let C be a closed subset of X. Let $\mathscr{C}$ be an open cover of C. Then $\mathscr{D}=\mathscr{C}\cup\{X-C\}$ is an open cover of X. By hypothesis, there exists a countable subset $\mathscr{C}'\subset\mathscr{C}$ such that $X=(X-C)\cup(\cup\text{cl}\{V\mid V\in\mathscr{C}'\})$. Thus $C\subset\cup\{\text{cl}\ V\mid V\in\mathscr{C}'\}$.

THEOREM 2.5. *Let (X,τ) be a HP-space. Then the following are equivalent:*

(i) X is α-minimal HP.

(ii) Given a closed subset C of X, a cover $\mathscr{C}$ of $X-C$ consisting of open or regular-open subsets of X and an open nbd V of C, there exists a countable subfamily $\mathscr{C}'\subset\mathscr{C}$ such that $X=V\cup(\cup\{cl\ U\mid U\in\mathscr{C}'\})$.

(iii) Given a closed subset C of X and a cover $\mathscr{C}$ of C consisting of open or

regular-open subsets of X, there exist a countable subfamily $\mathscr{C}'\subset\mathscr{C}$ *such that* $\mathscr{C}\subset\bigcup\{cl\, U\mid U\in\mathscr{C}'\}$.

(*iv*) *For each closed subset* $A(\subset X)$ *and each collection* $\mathscr{F}=\{F_\alpha\mid\alpha\in\Delta\}$ *of nonempty closed or regular-closed subsets of X such that* $(\bigcap\{F\mid F\in\mathscr{F}\})\cap A=\phi$, *there exists a countable subset* $\mathscr{F}'(\subset\mathscr{F})$ *such that* $(\bigcap\{int\, F\mid F\in\mathscr{F}'\})\cap A=\phi$.

(*v*) *For each closed subset* $A(\subset X)$ *and each collection* $\mathscr{F}=\{F_\alpha\mid\alpha\in\Delta\}$ *of nonempty closed or regular-closed subsets of X, if each countable subset* $\mathscr{F}'(\subset\mathscr{F})$ *has the property that* $(\bigcap\{int\, F\mid F\in\mathscr{F}'\})\cap A\neq\phi$, *then* $(\bigcap\{F\mid F\in\mathscr{F}\})\cap A\neq\phi$.

(*vi*) *For each closed subset* $A(\subset X)$ *and each σ-filter base* $\mathscr{F}=\{A_\alpha\mid\alpha\in\Delta\}$ *in A, there exists* $a\in A$ *such that* $\mathscr{F}$ *r-accumulates to a.*

(*vii*) *Each σ-filter base in X is r-accumulant convergent.*

PROOF. (i)⇒(ii). $C\subset V$ and $X-V$ is a closed subset of X such that $X-C\supset X-V$ so that $\mathscr{C}$ is an open cover of $X-V$. Hence there exists a countable subset $\mathscr{C}'\subset\mathscr{C}$ such that $X-V\subset\bigcup\{\text{cl}\, U\mid U\in\mathscr{C}'\}$. Thus $X=V\cup(\bigcup\{\text{cl}\, U\mid U\in\mathscr{C}'\})$.

(ii)⇒(iii). Given a closed subset C of X and a cover $\mathscr{C}$ of C consisting of open or regular-open subsets of X, $X-C$ is an open nbd of the closed set $X-\bigcup\{U\mid U\in\mathscr{C}\}$ and $\mathscr{C}$ is a cover of $\bigcup\{U\mid U\in\mathscr{C}\}$ consisting of open or regular-open subsets of X. Then, by (ii), $X=(X-C)\cup(\bigcup\{\text{cl}\, U\mid U\in\mathscr{C}'\})$ where $\mathscr{C}'$ is a countable subset of $\mathscr{C}$. Hence $C\subset\bigcup\{\text{cl}\, U\mid U\in\mathscr{C}'\}$.

(iii)⇒(iv). If $(\bigcap\mathscr{F})\cap A=\phi$, then $A\subset\bigcup\{X-F\mid F\in\mathscr{F}\}$ and $\{X-F\mid F\in\mathscr{F}\}$ is a family of open or regular-open subsets of X. By (iii), there exists a countable subset $\mathscr{F}'\subset\mathscr{F}$ such that $A\subset\bigcup\{\text{cl}(X-F)\mid F\in\mathscr{F}'\}=\bigcup\{X-\text{int}\, F\mid F\in\mathscr{F}'\}$. Thus $(\bigcap\{\text{int}\, F\mid F\in\mathscr{F}'\})\cap A=\phi$.

(iv)⇔(v) is clear.

(iii)⇒(i) is clear from the definition of α-minimal *HP*-spaces.

(iv)⇒(iii). Let $\mathscr{C}$ be a cover of A consisting of either open or regular-open subsets of X. Then $A\cap(\bigcap\{X-U\mid U\in\mathscr{C}\})=\phi$. By hypothesis, there exists a countable subset $\mathscr{C}'\subset\mathscr{C}$ such that $A\cap(\bigcap\{\text{int}\,(X-U)\mid U\in\mathscr{C}'\})=\phi$ so that $A\cap(\bigcap\{(X-\text{cl}\, U)\mid U\in\mathscr{C}'\})=\phi$. Hence $A\subset\bigcup\{\text{cl}\, U\mid U\in\mathscr{C}'\}$.

(i)⇒(vi). Suppose there exists a σ-filter base $\mathscr{F}$ in A such that $\mathscr{F}$ does not r-accumulate to a for all $a\in A$. (Notice that the members of $\mathscr{F}$ are contained in A). Then for each $a\in A$, there exists an open set U_a and some $F_a\in\mathscr{F}$ such that $F_a\cap\text{cl}\, U_a=\phi$. The collection $\{U_a\mid a\in A\}$ is an open cover of A; so that, by (i), there exists a countable subset $B\subset A$ such that $A\subset\bigcup\{\text{cl}\, U_a\mid a\in B\}$. Let $F_b\in\mathscr{F}$ such that $F_b\subset\bigcap\{F_a\mid a\in B\}$. Since $F_b\neq\phi$, there exists an element $c\in B$

such that $F_b \cap \text{cl}\, U_c \neq \phi$. Since $F_b \subset F_c$, $F_c \cap \text{cl}\, U_c \neq \phi$ which is a contradiction.

(vi)$\Rightarrow$(v). Suppose that there exist a closed set $A \subset X$ and a collection $\mathscr{F}$ of closed or regular-closed subsets of X such that $(\cap\{\text{int}\, F \mid F \in \mathscr{F}'\}) \cap A \neq \phi$ for every countable subset $\mathscr{F}'(\subset \mathscr{F})$ but $(\cap\{F \mid F \in \mathscr{F}\}) \cap A = \phi$. Clearly sets of the form $(\text{int}\, F) \cap A$ together with $(\cap\{\text{int}\, F \mid F \in \mathscr{F}'\}) \cap A$ for all possible countable subsets $\mathscr{F}' \subset \mathscr{F}$ form a σ-filter base $\mathscr{G}$ in A. By hypothesis (vi), $\mathscr{G}$ r-accumulates to some point $a \in A$. Thus for each open nbd U_a of a and each set int F with $F \in \mathscr{F}$, $((\text{int}\, F) \cap A) \cap \text{cl}\, U_a \neq \phi$. Since $(\text{int}\, F) \cap A \neq \phi$, $F \cap A \neq \phi$ for each $F \in \mathscr{F}$. Since $(\cap\{F \mid F \in \mathscr{F}\}) \cap A = \phi$, there exists $F_a \in \mathscr{F}$ such that $a \notin F_a$. Hence $a \in X - F_a \subset X - \text{int}\, F_a = \text{cl}(X - F_a)$. $X - F_a$ is an open nbd of a such that $\text{cl}(X - F_a) \cap \text{int}\, F_a = \phi$, a contradiction. Thus $(\cap\{F \mid F \in \mathscr{F}\}) \cap A \neq \phi$.

(i)$\Rightarrow$(vii). Let $\mathscr{F}$ be a σ-filter base. Let the set of all r-accumulation points of $\mathscr{F}$ be C. Let V be an open subset of X such that $V \supset C$. If $a \in X - V$, then there exists an open nbd U_a of a and $F_a \in \mathscr{F}$ such that $(\text{cl}\, U_a) \cap F_a = \phi$. Now $\{U_a \mid a \in X - V\}$ is an open cover of $X - V$ which is closed. Hence there exists a countable subset $B \subset X - V$ such that $X - V \subset \cup\{\text{cl}\, U_a \mid a \in B\}$. Clearly there exists $F \in \mathscr{F}$ such that $F \subset \cap\{F_a \mid a \in B\}$. Further $F \cap (\cup\{\text{cl}\, U_a \mid a \in B\}) = \phi$ so that $F \cap (X - V) = \phi$ and $F \subset V$. Thus $\mathscr{F}$ is r-accumulant convergent.

(vii)$\Rightarrow$(vi). Let A be a closed subset of X. Let $\mathscr{F}$ be a σ-filter in A. Suppose $\mathscr{F}$ r-accumulates to a for no $a \in A$; then the r-accumulant of $\mathscr{F} \subset X - A$ which is open. Thus by hypothesis in (vii), $X - A \supset F$ for some $F \in \mathscr{F}$, a contradiction.

THEOREM 2.6. *The following are equivalent for a HP-space X*:

(*i*) *X is α-minimal HP.*

(*ii*) *X is HP-closed and rim-α.*

PROOF. That (i)$\Rightarrow$(ii) is obvious.

To prove (ii)$\Rightarrow$(i) we use the criterion (iv) given in Theorem 2.4. Let $\mathscr{C}$ be an open cover of X and $U \in \mathscr{C}$. For each $x \in U$ there exists a rim-α open nbd V_x such that $x \in V_x \subset U$. Now $(\mathscr{C} - \{U\}) \cup \{\{V_x \mid x \in U\}\}$ is an open cover of X. Since X is HP-closed there exist countable subsets $\mathscr{C}' \subset \mathscr{C} - \{U\}$ and $B \subset U$ such that $X = (\cup\{\text{cl}\, V \mid V \in \mathscr{C}'\}) \cup (\cup\{\text{cl}\, V_x \mid x \in B\})$. Since $\text{cl}\, V_x - U$ is closed, $\text{cl}\, V_x - U \subset \text{cl}\, V_x - V_x$ and $\mathscr{C} - \{U\}$ is an open cover of $\text{cl}\, V_x - U$, there exists a countable subset $\mathscr{C}^x \subset \mathscr{C} - \{U\}$ such that $\text{cl}\, V_x - U \subset \cup\{\text{cl}\, V^x \mid V^x \in \mathscr{C}^x\}$. Clearly $\cup\{\text{cl}\, V_x \mid x \in B\} \subset \cup\{\text{cl}\, V^x \mid V^x \in \mathscr{C}^x$ and $x \in B\} \cup U$. Thus $X = U \cup (\cup\{\text{cl}\, V^x \mid V^x \in \mathscr{C}^x$

and $x \in B\}) \cup (\cup \{\mathrm{cl}\, V \mid V \in \mathscr{C}'\})$. Notice that $\mathscr{C}^x$ and B are countable and hence X is α-minimal *HP*.

THEOREM 2.7. *If (X, τ) is α-minimal HP-space, then every continuous function $f: X \longrightarrow Y$ into a Hausdorff P-space is closed.*

PROOF. Let B be closed in X and $p \notin f(B)$. Then for each $y \in f(B)$, there exists an open set U_y in Y such that $y \in U_y$ and $p \notin \mathrm{cl}\, U_y$. Now $\{f^{-1}(U_y) \mid y \in f(B)\}$ is an open cover of B in Y. This collection has a countable almost subcover $\{f^{-1}(U_i) \mid 1 \leq i < \infty\}$ so that $f(B) \subset \cup \{\mathrm{cl}\, U_i \mid 1 \leq i < \infty\}$, $p \in X - \cup \{\mathrm{cl}\, U_i \mid 1 \leq i < \infty\} \subset X - f(B)$ and hence $f(B)$ is closed.

Now we prove mapping characterizations of α-minimal *HP*-spaces.

THEOREM 2.8. *Let (X, τ) be a HP-space. Then the following are equivalent:*
(i) X is α-minimal HP.
(ii) Every continuous T_1P-image of X is μ-Lindelöf.
(iii) Every T_1P-quotient of X is μ-Lindelöf.
(iv) Every closed continuous T_1P-image of X is μ-Lindelöf.

PROOF. (i)$\Rightarrow$(ii). Every continuous image of α-minimal *HP*-space is α-minimal *HP* and hence by virtue of Theorem 2.4 (ii) above, it is μ-Lindelöf. If we relax the Hausdorff property to T_1 property, the image space will continue to be μ-Lindelöf without being Hausdorff.

Clearly (ii)$\Rightarrow$(iii)$\Rightarrow$(iv).

(iv)$\Rightarrow$(i). If X is not α-minimal *HP*, there exists an open σ-filter base $\mathscr{F}$ in X with $A = \cap \{\mathrm{cl}\, U \mid U \in \mathscr{F}\}$ while $\mathscr{F}$ does not converge to A (Refer Theorem 2.4 (ii) above). We can assume without loss of generality that the interior of A is empty and $U \cap A = \phi$ for each $U \in \mathscr{F}$. If we identify the points of A, what results is a T_1P-quotient of X viz., Z with the quotient map h which is a closed continuous map of X onto Z, while Z is not μ-Lindelöf.

Finally let us give a result relating to seminormal *HP*-closed spaces. Indeed we have

THEOREM 2.9. *In the class of seminormal HP-spaces, a space is HP-closed if and only if it is α-minimal HP.*

3. Examples

It is clear from the definition that every Lindelöf *HP*-space is α-minimal *HP*. Further every α-minimal *HP*-space is minimal *HP*. This is so, because every open σ-filter base converges to its adherence (Theorem 2.4 (ii)); thus every open σ-filter base with unique point of adherence converges to its adherence; hence the space is minimal *HP* [4, Theorem 2.6 (iii)]. But there are α-minimal *HP*-spaces which are not Lindelöf and minimal *HP*-spaces which are not α-minimal *HP*.

EXAMPLE 3.1. Let $A=[0,\Omega[$ and $B=A\cup\{\Omega\}=[0,\Omega]$. Let $X=(A\times B)\cup\{\pi\}$. Let us define a topology τ on X as follows. Let each point in $A\times A$ be open. Partition A into uncountably many uncountable equivalence classes $\{A_\alpha|\alpha\in A\}$. $A_\alpha\cap A_\beta=\phi$ if $\alpha\neq\beta$ and $\cup\{A_\alpha|\alpha\in A\}=A$. Let us write $F_\alpha=[\alpha,\Omega[$ where α is an element of $[0,\Omega[$ and $F_\alpha\cup[0,\alpha)=A$. Let a nbd system for the point $(\alpha,\Omega)\in A\times\{\Omega\}$ $(\subset A\times B)$ be sets of the form

$$V((\alpha,\Omega)\ ;\beta)=\{(\alpha,\Omega)\}\cup(\{\alpha\}\times F_\beta)\cup(F_\beta\times A_\alpha)$$

for some $\beta\in A$. Let $K(\alpha)=(\{\alpha\}\times A)\cup(A\times A_\alpha)$ and $L(\beta)=(\cup\{K(\alpha)\,|0\leqq\alpha<\beta\})\cup(A\times\{\Omega\})$. Let a nbd system for the point π be sets of the form

$$V(\pi\ ;\beta)=(X-L(\beta))\cup\{\pi\}$$

for some $\beta\in A$.

Thus we get a space (X,τ) which, by construction, is a *HP*-space. Further this space is not Lindelöf because the closed subset $\{(\alpha,\Omega)\,|\alpha\in A\}$ is discrete and uncountable and hence not Lindelöf.

Now let us prove that (X,τ) is *HP*-closed and rim-α. Then by Theorem 2.6, (X,τ) will be α-minimal *HP*.

First let us show that (X,τ) is rim-α. If $p\in A\times A\subset X$ a basic τ-open nbd system at p is $\{\{p\}\}$. $\mathrm{cl}(\{p\})-\{p\}=\phi$. Thus the condition for rim-α is trivially satisfied at p. Consider $(\xi,\Omega)\in X$ where $\xi\in A$. A typical basic τ-open nbd of (ξ,Ω) is of the form $V((\xi,\Omega)\ ;\beta)=\{(\xi,\Omega)\}\cup(\{\xi\}\times F_\beta)\cup(F_\beta\times A_\xi)$ for some $\beta\in A$. $\mathrm{cl}(V((\xi,\Omega)\ ;\beta))=V((\xi,\Omega)\ ;\beta)\cup(\cup\{(\alpha,\Omega)|\alpha\in F_\beta\})\cup\{\pi\}$. Let Y be any subset of $(\cup\{(\alpha,\Omega)\,|\alpha\in F_\beta\})\cup\{\pi\}$. Let $(\xi_1,\Omega)\in Y$. A typical open nbd of (ξ_1,Ω) is of the form $V((\xi_1,\Omega)\ ;\gamma)=\{(\xi_1,\Omega)\}\cup(\{\xi_1\}\times F_\gamma)\cup(F_\gamma\times A_{\xi_1})$ for some $\gamma\in A$. $\mathrm{cl}(V((\xi_1,\Omega)\ ;\gamma))\supset(F_\gamma\times\{\Omega\})\cup\{\pi\}$ which covers all but countably many members of Y. Thus the condition for rim-α is satisfied at (ξ,Ω) also. A typical basic open nbd $V(\pi\ ;\delta)$ is of the form $(X-L(\delta))\cup\{\pi\}$ for some $\delta\in A$. $\mathrm{cl}\,V(\pi\ ;\delta)-$

$V(\pi ; \delta)=F_\delta \times \{\Omega\}$. As we have already seen, every open cover (open in X) of every subset of $F_\delta \times \{\Omega\}$ admits a countable almost subcover. Thus the condition for rim-α is satisfied at π also.

Now let us prove that the space is *HP*-closed. Let $\mathscr{U}$ be an open cover of X. Let $V(\pi ; \beta) (\in \mathscr{U})$ be an open basic nbd of π. Then $X-\mathrm{cl}\, V(\pi ; \beta) \subset \bigcup \{\{(\alpha, \Omega)\} \cup (\{\alpha\} \times A) \cup (A \times A_\alpha) \mid \alpha \in [0, \beta)\}$. Let γ be an ordinal such that $0 \leqq \gamma < \beta$. Then there exists $\delta(\gamma)$ such that $(\gamma) \times B$ is contained in $V((\gamma, \Omega) ; \delta(\gamma)) \in \mathscr{U}$ except for countably many points. Moreover $F_{\delta(\gamma)} \times A_\gamma$ is contained in $V((\gamma, \Omega) ; \delta(\gamma))$. Let ξ be an ordinal such that $\xi \neq \gamma$ and $0 \leqq \xi < \delta(\gamma)$. Then there exist open nbds of (ξ, Ω), namely, $V((\xi, \Omega) ; \delta(\xi)) \in \mathscr{U}$ such that if we write

$$V(\gamma)=V((\gamma, \Omega) ; \delta(\gamma)) \cup (\bigcup \{V((\xi, \Omega) ; \delta(\xi)) \mid 0 \leqq \xi < \delta(\gamma) \text{ and } \xi \neq \gamma\})$$

then $V(\gamma)$ covers $(\{\gamma\} \times B) \cup (A \times A_\gamma)$, all but countably many elements in it. Note that the collection of sets in $V(\gamma)$ is countable and then we allow γ to vary in $[0, \beta)$. Thus we see that (X, τ) is *HP*-closed.

EXAMPLE 3.2. This example is given in [4, Example 2.8]. We have proved that this space is minimal *HP*. We claim that the space is not α-minimal *HP*. Indeed we take a typical basic open nbd of 0, say $V(P, 0)$, then $\mathrm{cl}\, V(P, 0)-V(P, 0)=M=\{x_b \mid x \in B_1-P\}$. Indeed M is an uncountable closed discrete subspace such that it fails to satisfy the rim-α condition (at 0). Thus X is not α-minimal *HP*.

EXAMPLE 3.3. In [5] Viglino introduced the concept of C-compact spaces. The definition of such spaces runs parallel to our definition of α-minimal *HP*-spaces. Now let us give a relatively simple example to show that the classes of C-compact and α-minimal *HP* spaces are distinct. Let $X=[1, \omega)$ be a discrete space. Let $X^*=[1, \omega]$ be its one-point compactification. Thus X^* is C-compact. But it is not even a P-space and hence not α-minimal *HP*. Again if $[1, \omega]$ is given the discrete topology, the resulting space is a countable *HP*-space so that it is α-minimal *HP* (indeed, Lindelöf). But this space is not minimal Hausdorff and hence not C-compact.

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