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TOPOLOGICAL PROJECTIVE PLANES

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1. Introduction

A topological projective plane is a projective plane in which the space of points $\mathscr{P}$ and the space of lines $\mathscr{L}$ are endowed with topologies $\tau_{\mathscr{P}}$ and $\tau_{\mathscr{L}}$ respectively such that the operations of joining and intersecting are continuous in both variables. We obtain a description of the topologies $\tau_{\mathscr{P}}$ and $\tau_{\mathscr{L}}$ in terms of the topology τ of the associated ternary ring by using convergence of nets and deduce that in the case of a topological pappian plane the space of points is homeomorphic to the space of lines. We also obtain another description of the topologies $\tau_{\mathscr{P}}$ and $\tau_{\mathscr{L}}$ by specifying the basic open sets. Using this we derive a necessary and sufficient condition for a locally compact topological ternary ring to coordinatise a compact projective plane. We assume the coordinatization of a projective plane given in [2]. All the topologies considered are assumed to be Hausdorff.

THEOREM 1.1 [3]. *The ternary ring $(S.T)$ of a topological projective plane is a topological ternary ring with a topology τ which is homeomorphic to an affine ray of points.*

THEOREM 1.2 [2]. *The set of points of an affine plane is homeomorphic to the product of an affine ray with itself.*

For topological terminologies we refer [1].

2. Description of $\tau_{\mathscr{P}}$ and $\tau_{\mathscr{L}}$ in terms of τ.

THEOREM 2.1 *Let (S, T, τ) be a topological ternary ring coordinatising a topological projective plane. Then the convergence schemes in $(\mathscr{P}, \tau_{\mathscr{P}})$ and $(\mathscr{L}, \tau_{\mathscr{L}})$ are as follows.*

(i) *A net $(x_\alpha, y_\alpha) \to (x, y)$ iff $x_\alpha \to x$ and $y_\alpha \to y$.*

(ii) *$(x_\alpha, y_\alpha) \to (m)$ iff $x_\alpha^l \to 0$ and $m_\alpha \to m$ where x_α^l is the left inverse of x_α in S and $y_\alpha = m_\alpha x_\alpha$.*

(iii) *(x_α, y_α), $x_\alpha \neq 0$, $\to (\infty)$ iff $y_\alpha^l \to 0$ and $m_\alpha^l \to 0$ where $y_\alpha = m_\alpha x_\alpha$.*

(iv) $(0,y_\alpha)\to(\infty)$ *iff* $y^l_\alpha\to 0$.

(v) $(m_\alpha)\to(m)$ *iff* $m_\alpha\to m$.

(vi) $(m_\alpha)\to(\infty)$ *iff* $m^l_\alpha\to 0$.

(vii) *A net of lines* $[m_\alpha, c_\alpha]$ *converges to the line* $[m,c]$ *iff* $m_\alpha\to m$ *and* $c_\alpha\to c$.

(viii) $[m_\alpha, c_\alpha]\to[k]$ *iff* $m^l_\alpha\to 0$ *and* $x_\alpha\to k$ *where* $T(m_\alpha, x_\alpha, c_\alpha)=0$.

(ix) $[m_\alpha, c_\alpha]$, $m_\alpha\neq 0, \to[\infty]$ *iff* $m^l_\alpha\to 0$ and $x^l_\alpha\to 0$.

(x) $[0, c_\alpha]\to\infty$ *iff* $c^l_\alpha\to 0$.

(xi) $[k_\alpha]\to[k]$ *iff* $k_\alpha\to k$.

(xii) $[k_\alpha]\to[\infty]$ *iff* $k^l_\alpha\to 0$.

PROOF. Since (S,τ) is homeomorphic to an affine ray of points and the product space $S\times S$ is homeomorphic to the set of points of an affine plane we get (i) and (v). We prove (ii) and the proof for the remaining cases is similar. Let $(x_\alpha, y_\alpha)\to(m)$. We may assume that $x_\alpha\neq 0$ for all α. Since $y_\alpha=m_\alpha x_\alpha$, the point (x_α, y_α) lies on the line $[m_\alpha, 0]$ and it follows from the continuity of joining that the net of lines $[m_\alpha, 0]\to[m,0]$. Hence $(m_\alpha)\to(m)$ so that $m_\alpha\to m$. Also $[x_\alpha]\to[\infty]$ and hence $[x_\alpha]\cap[0,1]=(x_\alpha,1)\to(0)$ so that $x^l_\alpha\to 0$. Converse follows by reversing the above steps.

The ternary ring of a Pappian plane is a field F and a point is represented by a family of ordered triples $(a\lambda, b\lambda, c\lambda)$ where a,b,c are fixed but not all zero and $\lambda\neq 0$. A line also is represented by a family of ordered triples $[\mu l, \mu m, \mu n]$ where l,m,n are not all zero and $\mu\neq 0$.

Further $(x,y,z)\in[l,m,n]$ iff $lx+my+nz=0$.

The function $(x,y)\to(x,y,1)$

$(m)\to(1,m,0)$

and $(\infty)\to(0,1,0)$ is an incidence preserving map between the two labellings of the set of points. In this case the convergence scheme described in theorem 2.1 leads to the following.

THEOREM 2.2. *In a topological Pappian plane coordinatised by a field F a net of points* P_α *converges to P iff there exist representations* $P_\alpha^*=(x_\alpha, y_\alpha, z_\alpha)$ *for* P_α *and* $P^*=(x,y,z)$ *for P such that* $P_\alpha^*\to P^*$ *in* F^3.

A similar result is true for the space of lines of a topological Pappian plane and hence we have

THEOREM 2.3 *In a topological Pappian plane the space of points is homeomorphic to the space of lines.*

In the following theorem we specify a basis for the spaces $(\mathscr{P}, \tau_{\mathscr{P}})$ and $(\mathscr{L}, \tau_{\mathscr{L}})$ in terms of the open sets in the associated topological ternary ring (S, T, τ).

THEOREM 2.4 *Let (S, T, τ) be a topological ternary ring coordinatizing a topological projective plane. Then the basic open sets in $(\mathscr{P}, \tau_{\mathscr{P}})$ and $(\mathscr{L}, \tau_{\mathscr{L}})$ are as follows:*

(i) *A basic open set containing (x,y) is of the form $U\times V$ where U is an open set containing x and V is an open set containing y in S.*

(ii) *A basic open set containing the point (m) is obtained as follows: Let U be an open set containing 0 and V be an open set containing m in S.*
Let $N_{U,V}=\{(x,y)\,|\,x^l\in U$ and $w\in V$ where $y=wx\}\ \cup\{(w)\,|\,w\in V\}$.
Then $N_{U,V}$ is a basic open set containing (m).

(iii) *A basic open set containing the point (∞) is obtained as follows: Let U be an open set containing 0 in S.*
Let $N_U=\{(x,y)\,|\,y^l\in U$ and $w^l\in U$ where $y=wx\}\ \cup\{(w)\,|\,w^l\in U\}\ \cup\{(\infty)\}$.
Then N_U is a basic open set containing (∞).

(iv) *A basic open set containing the line $[m,c]$ is obtained as follows: Let U be an open set containing m and V be an open set containing c in S. Then $\{[s,t]\,|\,s\in U,\ t\in V\}$ is a basic open set containing $[m,c]$.*

(v) *A basic open set containing the line $[k]$ is obtained as follows: Let U be an open set containing 0 and V be an open set containing k in S.*
Let $N'_{U,V}=\{[s,t]\,|\,s^l\in U$ and $r\in V$ where r is given by $T(s,r,t)=0\}\ \cup\{[r]\,|\,r\in V\}$.
Then $N'_{U,V}$ is a basic open set containing $[k]$,

(vi) *A basic open set containing the line $[\infty]$ is obtained as follows: Let U be an open set containing 0 in S.*
Let $N'_U=\{[s,t]\,|\,t^l\in U$ and $r^l\in U$ where r is given by $T(s,r,t)=0\}\ \cup\{[r]\,|\,r^l\in U\}$ $\cup\{[\infty]\}$.
Then N'_U is a basic open set containing $[\infty]$.

PROOF. It is easy to show that the family of sets described in (i), (ii) and (iii) forms a basis for a topology on $\mathscr{P}$ and the convergence scheme with respect to this topology coincides with the convergence scheme described in theorem 2.1.

3. Compact projective planes

In a topological projective plane, $(\mathscr{P}, \tau_{\mathscr{P}})$ is compact iff $(\mathscr{L}, \tau_{\mathscr{L}})$ is compact. Such a projective plane is called a compact projective plane. In this case the associated ternary ring is locally compact. Now, let (S, T, τ) be a locally compact topological ternary ring coordinatising a compact projective plane. We denote by $S \cup \{\infty\}$ the one-point compactification of (S, τ). Clearly $f: S \cup \{\infty\} \rightarrow [\infty]$ defined by $f(m)=(m)$ and $f(\infty)=(\infty)$ is a homeomorphism where $[\infty]$ is given the relative topology of $(\mathscr{P}, \tau_{\mathscr{P}})$. Hence it follows from theorem 2.1 that a net x_α in S converges to the ideal point ∞ in the one-point compactification of S iff $x_\alpha^l \rightarrow 0$.

THEOREM 3.1 *Let (S, T, τ) be a locally compact topological ternary ring coordinatizing a compact projective plane. Then the basic open sets in $(\mathscr{P}, \tau_{\mathscr{P}})$ and $(\mathscr{L}, \tau_{\mathscr{L}})$ are as follows.*

(i) *A basic open set containing (x,y) is of the form $U \times V$ where U is an open set containing x and V is an open set containing y in S.*

(ii) *Let C be a compact subset of $S \times S$ and V an open set containing m in S. Let $B_{C,V} = \{(x,y) \mid (x,y) \notin C$ and $y = wx$ for some $w \in V\} \cup \{(w) \mid w \in V\}$. Then $B_{C,V}$ is a basic open set containing (m).*

(iii) *A basic open set containing (∞) is obtained as in (ii) by replacing V by an open set containing the ideal point ∞ in the one point compactification of S.*

(iv) *Let U be an open set containing m and V be an open set containing c in S. Then $\{[s,t] \mid s \in U,\ t \in V\}$ is a basic open set containing $[m,c]$.*

(v) *Let C be a compact subset of $S \times S$ and V be an open set containing k in S. Let $B'_{C,V} = \{[s,t] \mid [s,t] \notin C$ and $r \in V$ where $T(s,r,t)=0\} \cup \{[r] \mid r \in V\}$. Then $B'_{C,V}$ is a basic open set containing $[k]$.*

(vi) *A basic open set containing the line $[\infty]$ is obtained as in (v) by replacing V by an open set containing the ideal point ∞ in the one point compactification of S.*

PROOF. The family of sets described in (i), (ii) and (iii) forms a basis for a topology on $\mathscr{P}$ and this topology is the same as the topology described in theorem 2.4.

THEOREM 3.2 *A topological projective plane coordinatised by a non-discrete locally compact ternary ring (S, T, τ) is compact iff each ray of points is homeomorphic to the one-point compactification of S.*

PROOF. Suppose that each ray of points is homeomorphic to the one point compactification of S. Let $\{V_i\}$ be an open cover for $\mathscr{P}$ by basic open sets described in theorem 3.1. Since the ray of points $[\infty]$ is compact, it can be covered by a finite number of open sets, say $V_1, V_2, \cdots, V_n$. Now $\mathscr{P}-(V_1 \cup V_2 \cup \cdots V_n)$ is a compact subset of the affine plane $\mathscr{P}-[\infty]$ and hence can be covered by a finite number of V_i. Hence $(\mathscr{P}, \tau_{\mathscr{P}})$ is compact. The converse is trivial.

COROLLARY 3.3 *A topological projective plane coordinatised by a locally compact non-discrete topological ternary ring (S, T, τ) is compact iff (S, τ) satisfies the following condition (A).*

(A): A net x_α in S converges to the ideal point ∞ in the one-point compactification of S iff $x_\alpha^l \to 0$.

THEOREM 3.4 *A topological projective plane coordinatised by a locally compact non-discrete topological ternary ring (S, T, τ) is compact, iff (S, τ) satisfies the following condition (B).*

(B): If x_α and y_α are nets in S converging to the ideal point ∞ in the one-point compactification of S then the net $x_\alpha y_\alpha$ also converges to ∞.

PROOF. Let $x_\alpha \to \infty$. Suppose x_α^l has a convergent subnet converging to an element $y \neq 0$ in S. Let $y = x^l$ where $x \in S$. Then the corresponding subnet of x_α converges to x which is a contradiction. Also if x_α has a convergent subnet converging to ∞ then by (B) the constant net 1 converges to ∞ which is again a contradiction. Thus every convergent subnet of x_α^l converges to 0 and hence x_α^l converges to 0. Similarly, if x_α^l converges to 0 then x_α converges to ∞. Thus (S, τ) satisfies condition (A) of Corollary 3.3 and hence the plane is compact.

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REFERENCES

[1] Kelley. General Topology, Van Nostrand (1955).

[2] Pickert, G., Projective Ebenen, Springer (1955).

[3] Salzmann, H., Advances in Mathematics, Topological Planes, Vol. 2. Fascile 1 Academic Press (1967).

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