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# A NOTE ON FUNCTIONS SATISFYING Re$\{f(z)/z\}>\alpha$

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## 1. Introduction

Let $S(\alpha)$ denote the class functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1.1}$$

which are analytic in the unit disk $U=\{z: |z|<1\}$ and satisfy the following condition

$$\mathrm{Re}\left\{\frac{f(z)}{z}\right\}>\alpha \qquad (z\in U) \tag{1.2}$$

for some $\alpha (0\leqq\alpha<1)$.

This class $S(\alpha)$ was studied by Goel [5] and Chen [3]. In particular, The class $S(0)$ was studied by Goel [6] and Yamaguchi [15].

We introduce the class $S_{G(B)}(\alpha)$ of funtions $f(z)\in S(\alpha)$ such that $G(z)\in S(\beta)$, where $G(z)=\Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}f(z)$ and $D_z^{\lambda}f(z)$ means the fractional derivative of order $\lambda$ of $f(z)$. The object of the present paper is to show some distortion theorems for $f(z)$ belonging the class $S_{G(B)}(\alpha)$.

Now, there are many definitions of the fractional calculus, that is, the fractional integrals and the fractional derivatives. We find it convenient to restrict ourselves to the following definitions of the fractional calculus used recently by Owa [9].

DEFINTION 1. The fractional integral of order $\lambda$ is defined by

$$D_z^{-\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_0^z\frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\lambda}}, \tag{1.3}$$

where $\lambda>0$, $f(z)$ is an analytic function in a simply connected region of the $z$-plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

DEFINITION 2. The fractional derivative of order $\lambda$ is defined by

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(1.4) $$D_z^\lambda f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^\lambda},$$

where $0<\lambda<1$, $f(z)$ is an analytic function in a simply connected region of the $z$-plane containing the origin and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\lambda)$ is defined by

(1.5) $$D_z^{n+\lambda} f(z)=\frac{d^n}{dz^n} D_z^\lambda f(z),$$

where $0<\lambda<1$ and $n\in N\cup\{0\}$.

REMARK. For other definitions of the fractional calculus, see Agarwal [1], Al-Salam [2], Erdélyi, Magnus, Oberhettinger and Tricomi [4], Nishimoto [7], Osler [8], Ross [11], Saigo [12], Sneddon [13] and Srivastava and Buschman [14].

**2. The class $S_{G(\beta)}(\alpha)$**

For the function $f(z)$ of the form (1.1), with a simple computation, we get

(2.1) $$G(z)=\Gamma(2-\lambda)z^\lambda D_z^\lambda f(z) =z+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}a_n z^n$$

for $0<\lambda<1$.

THEOREM 1. *Let $n\geqq 2$, $(n-1)/n\leqq\alpha<1$, $0\leqq\beta<1$ and $n(1-\alpha)\leqq 1-\beta$. Then there exists the function of $S(\alpha)$ defined by*

(2.2) $$f(z)=z+a_n z^n$$

*such that $G(z)\in S(\beta)$*

PROOF. Let the function $f(z)$ defined by (2.2) belong to the class $S(\alpha)$. Then we can see that

(2.3) $$\mathrm{Re}(a_n z^{n-1})>\alpha-1$$

and

(2.4) $$G(z)=z+\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}a_n z^{n-1}.$$

Consequently, we obtain

(2.5) $$\mathrm{Re}\left\{\frac{G(z)}{z}\right\}=1+\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}Re(a_n z^{n-1})$$

$$>1+\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}(\alpha-1)$$
$$\geqq\beta$$

for $(n-1)/n\leqq\alpha<1$ and $n(1-\alpha)\leqq 1-\beta$. This shows that $G(z)\in S(\beta)$.

In view of Theorem 1, let $S_{G(\beta)}(\alpha)$ denote class of functions $f(z)\in S(\alpha)$ of the form (1.1) such that $G(z)\in S(\beta)$.

We need the following lemma by Chen [3].

LEMMA 1. *Let the function $f(z)$ of the form* (1.1) *belong to the class* $S(\alpha)$. *Then we have*

$$\mathrm{Re}\{f'(z)\}\geqq\frac{1+(4\alpha-2)|z|+(2\alpha-1)|z|^2}{(1+|z|)^2} \tag{2.6}$$

*for* $0\leqq|z|<1/2$ *and*

$$\mathrm{Re}\{f'(z)\}\geqq\frac{\alpha-2\alpha|z|^2+(2\alpha-1)|z|^4}{(1-|z|^2)^2} \tag{2.7}$$

*for* $1/2\leqq|z|<1$. *These results are sharp.*

Further we need the following lemma by Owa [10].

LEMMA 2. *Let the function $f(z)$ of the from* (1.1) *belong to the class* $S(\alpha)$. *Then we have*

$$\left|\frac{f(z)}{z}\right|\leqq\frac{1+(1-2\alpha)|z|}{1-|z|} \tag{2.8}$$

*and*

$$\mathrm{Re}\left\{\frac{f(z)}{z}\right\}\geqq\frac{1-(1-2\alpha)|z|}{1+|z|} \tag{2.9}$$

*for* $z\in U$. *The results are sharp.*

THEOREM 2. *Let the function $f(z)$ of the form* (1.1) *belong to the class* $S_{G(\beta)}(\alpha)$. *Then we have*

$$|D_z^\lambda f(z)/z^{1-\lambda}|\leqq\frac{1+(1-2\beta)|z|}{(1-|z|)\Gamma(2-\lambda)} \tag{2.10}$$

*and*

$$\mathrm{Re}\{D_z^\lambda f(z)/z^{1-\lambda}\}\geqq\frac{1-(1-2\beta)|z|}{(1+|z|)\Gamma(2-\lambda)} \tag{2.11}$$

*for* $0<\lambda<1$ *and* $z\in U$. *The results are sharp.*

PROOF. Since $G(z)$ defined by (2.1) is in the class $S(\beta)$, in virtue of (2.8), we can show that

$$|\Gamma(2-\lambda)z^{\lambda-1}D_z^\lambda f(z)|\leqq\frac{1+(1-2\beta)|z|}{1-|z|} \tag{2.12}$$

from which (2.10) follows at once. Further the second estimate (2.11) of the theorem follows from (2.9).

Finally, the results of the theorem are sharp for the function $f(z)$ defined by

$$D_z^\lambda f(z)=\frac{\{1+(1-2\beta)z\}z^{1-\lambda}}{(1-z)\Gamma(2-\lambda)}. \tag{2.13}$$

COROLLARY 1. *Let the function $f(z)$ of the form* (1.1) *belong to the class $S_{G(0)}(\alpha)$. Then we have*

$$|D_z^\lambda f(z)/z^{1-\lambda}|\leqq\frac{1+|z|}{(1-|z|)\Gamma(2-\lambda)} \tag{2.14}$$

*and*

$$\mathrm{Re}\{D_z^\lambda f(z)/z^{1-\lambda}\}\geqq\frac{1-|z|}{(1+|z|)\Gamma(2-\lambda)} \tag{2.15}$$

*for $0<\lambda<1$ and $z\in U$. The results are sharp.*

THEOREM 3. *Let the function $f(z)$ of the frorm* (1.1) *belong to the class $S_{G(\beta)}(\alpha)$. Then we have*

$$\mathrm{Re}\{z^\lambda D_z^{1+\lambda}f(z)\}\geqq\frac{(1-\lambda)+2(2\beta-\beta\lambda-1)|z|+(1-\lambda)(1-2\beta)|z|^2}{(1+|z|)^2\Gamma(2-\lambda)} \tag{2.16}$$

*for $0<\lambda<1$ and $0\leqq|z|<1/2$, and*

$$\mathrm{Re}\{z^\lambda D_z^{1+\lambda}f(z)\} \geqq\frac{(\beta-\lambda)+2\lambda(1-\beta)|z|-2\beta(1-\lambda)|z|^2-2\lambda(1-\beta)|z|^3+(2\beta-1)(1-\lambda)|z|^4}{(1-|z|^2)^2\Gamma(2-\lambda)} \tag{2.17}$$

*for $0<\lambda<1$ and $1/2\leqq|z|<1$.*

PROOF. Since $G(z)$ defined by (2.1) is in the class $S(\beta)$, Lemma 1 implies that

$$\mathrm{Re}\{G'(z)\}\geqq\frac{1+(4\beta-2)|z|+(2\beta-1)|z|^2}{(1+|z|)^2} \tag{2.18}$$

for $0\leqq|z|<1/2$ and

$$\mathrm{Re}\{G'(z)\}\geqq\frac{\beta-2\beta|z|^2+(2\beta-1)|z|^4}{(1-|z|^2)^2} \tag{2.19}$$

for $1/2\leqq|z|<1$. Hence, by using of (2.11) and (2.18), we get

$$\mathrm{Re}\{z^\lambda D_z^{1+\lambda}f(z)\}\geqq\frac{1+(4\beta-2)|z|+(2\beta-1)|z|^2}{(1+|z|)^2\Gamma(2-\lambda)}-\lambda\mathrm{Re}\{D_z^\lambda f(z)/z^{1-\lambda}\} \tag{2.20}$$

$$\geqq \frac{(1-\lambda)+2(2\beta-\beta\lambda-1)|z|+(1-\lambda)(1-2\beta)|z|^2}{(1+|z|)^2\Gamma(2-\lambda)}$$

for $0<\lambda<1$ and $0\leqq|z|<1/2$. Further, in virtue of (2.11) and (2.19),

$$\text{(2.21)}\quad \mathrm{Re}\{z^\lambda D_z^{1+\lambda} f(z)\}$$
$$\geqq \frac{\beta-2\beta|z|^2+(2\beta-1)|z|^4}{(1-|z|^2)^2\Gamma(2-\lambda)}-\lambda\mathrm{Re}\{D_z^\lambda f(z)/z^{1-\lambda}\}$$
$$\geqq \frac{(\beta-\lambda)+2\lambda(1-\beta)|z|-2\beta(1-\lambda)|z|^2-2\lambda(1-\beta)|z|^3+(2\beta-1)(1-\lambda)|z|^4}{(1-|z|^2)^2\Gamma(2-\lambda)}$$

for $0<\lambda<1$. $1/2\leq|z|<1$. Thus we have the theorem.

COROLLARY 2. *Let the function $f(z)$ of the form* (1.1) *belong to the class $S_{G(0)}(\alpha)$. Then we have*

$$\text{(2.22)}\qquad \mathrm{Re}\{z^\lambda D_z^{1+\lambda} f(z)\}\geqq \frac{(1-\lambda)-2|z|+(1-\lambda)|z|^2}{(1+|z|)^2\Gamma(2-\lambda)}$$

*for $0<\lambda<1$ and $0\leqq|z|<1/2$, and*

$$\text{(2.23)}\qquad \mathrm{Re}\{z^\lambda D_z^{1+\lambda} f(z)\}\geqq \frac{-\lambda+2\lambda|z|-2\lambda|z|^3-(1-\lambda)|z|^4}{(1-|z|^2)^2\Gamma(2-\lambda)}$$

*for $0<\lambda<1$ and $1/2\leqq|z|<1$.*

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