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A NOTE ON THE MODULAR GROUP RING OF A FINITE p-GROUP

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In [1] Johnson has proved when G is a group of prime power order p and K a field having only p-elements then G^*, the mod p-envelope of G has a group structure. It is interesting to note when G is a group of order 2 and K a field with more than 2 elements then G^*, the mod p-envelope of G has a semi-group structure with a non-trivial idempotent in it. Further in this case the number of elements in G^* is equal to the number of elements in the field K. For the sake of completeness we prove the following propositions. For definitions please refer [1].

PROPOSITION 1. *Let* $G=\langle g\,|\,g^2=1\rangle$ *be a group of order* 2 *and* $K=(0,1)$ *be a field with two elements, Then* $G^*=G$.

PROOF. Consider $U=\{0,1+g\}$

$G^*=1+U=\{1,g\}$.

Thus $G=G^*$ is a group.

PROPOSITION 2. *Let* $G=\langle g\,|\,g^2=1\rangle$ *be a group and* $K=(0,1,2,3,4)$ *(*K *is a field of* 5 *elements). Then* G^* *is a semi-group with order of* G^* *equal to* 5.

PROOF. Consider $U=\{0,1+4g,\ g+4,\ 2+3g,\ 3+2g\}$. $G^*=1+U=\{1,4g+2,$ $g,3+3g,\ 4+2g\}$. Clearly G^* is a semi-group with $3+3g$ an idempotent in G^*. Further the order of G^* is equal to 5.

THEOREM 3. *Let* $G=\langle g\,|\,g^2=1\rangle$ *be a group of order* 2. K *a field of* p*-elements,* $K=(0,1,2,\cdots,p-1)$. *Then* G^* *is a semi-group such that order of* $G^*=p$. *Further* $\left(\frac{p+1}{2}+\frac{p+1}{2}g\right)$ *is an idempotent in* G^*.

PROOF. $G^*=\left\{1,(p-1)g+2,\ 2g+(p-1),\ (p-2)g+3,\ 3g+p-2,\ \cdots\cdots,\ \frac{p+1}{2}+\frac{p+1}{2}g\right\}$. Clearly G^* is a semi-group with p-elements.

Further consider $\left(\frac{1+p}{2}+\frac{1+p}{2}g\right)^2=\left(\frac{1+p}{2}\right)^2(1+g)^2$

$$= \frac{1+p}{2} \cdot \frac{1+p}{2}(1+g)^2$$
$$= \frac{1+p}{2} \cdot \frac{1+p}{2} 2(1+g)$$
$$= \left(\frac{1+p}{2}\right)(1+p)(1+g)$$
$$p=0 (\mathrm{mod}\ p)$$
$$= \frac{1+p}{2}(1+g)$$

is an idempotent G^*.

It is essential to make the following remark.

REMARK. Let $G=\langle g \mid g^4=1\rangle$ be a group of order 4. $K=(0,1)$ be a field of 2 elements. KG be the group ring of G over K. Then G^* is a group of order 8.

PROOF. Consider $G^*=1+U=\{1, g, g^2, g^3, 1+g+g^2, 1+g+g^3, 1+g^2+g^3, g+g^2+g^3\}$

$$(g)^4=1,\ (g^3)^4=1,\ (g^2)^2=1$$
$$(1+g+g^2)^4=1 (1+g+g^3)^2=1$$
$$(1+g^2+g^3)^2=1 \text{ and } (g+g^2+g^3)^2=1.$$

Thus every element is of order 2 or 4.
Further G^* is generated by any one of the following sets

$$\langle g, 1+g+g^2\rangle$$
$$\langle g^3, 1+g^2+g\rangle$$
$$\langle g, 1+g+g^3\rangle$$
$$\langle g^3, 1+g+g^3\rangle$$
$$\langle g, 1+g^2+g^3\rangle$$
$$\langle g^3, 1+g^2+g^3\rangle$$
$$\langle g, g+g^2+g^3\rangle$$
$$\langle g^3, g+g^2+g^3\rangle$$
$$\langle 1+g+g^2,\ 1+g+g^3\rangle$$
$$\langle 1+g+g^2,\ g+g^2+g^3\rangle$$
$$\langle 1+g+g^3,\ g^3+g^2+1\rangle$$
$$\langle 1+g^2+g^3,\ g+g^2+g^3\rangle$$

Any one of the 12 pairs can generate G^*, $0(G^*)=8$.

From this remark we observe that if G is a group of prime power order say

p^n and K a field of p elements, then G^* still continues to be a group. When the order of G does not divide the order of the field G^* ceases to be a group. G^* has only a semi-group structure.

EXAMPLE. Let $G=\langle g \mid g^3=1\rangle$ be a group of order 3. $K=(0,1)$ be a field of 2 elements. Then G^* is a semi-group.
Consider $1+U=G^*=\{1, g, g^2, 1+g+g^2\}$. Clearly G^* is a semi-group with $1+g+g^2$ to be an idempotent in G^*.

From the above example we note that $0(G^*)=4$.

THEOREM 4. *Let G be a cyclic group of order p, p a prime and let $K=(0.1)$ be a field of two elements. Then G^* is a semi-group with $1+g+g^2+\cdots+g^{p-1}$ as an idempotent.*

PROOF. G^* contains elements such that the sum of the coefficients of those is one. Clearly G^* contains $1+g+g^2+\cdots+g^{p-1}$, the sum of the coefficients is p since p is prime and K is the field of characteristic 2 we have the coefficient of $1+g+g^2+\cdots+g^{p-1}$ is one.
Now consider

$$(1+g+g^2+\cdots+g^{p-1})^2=1+g+g^2+\cdots+g^{p-1}$$

is an idempotent.

Clearly as the sum of the coefficient is one of the set G^* is closed with respect to multiplication having a unit element.

We make the following remark about this Theorem. If the order of the group is not a prime but an even integer say $2n$ then obviously $1+g+\cdots+g^{2n}$ is not an idempotent as is evidenced by an example.

EXAMPLE. Let $G=\langle g \mid g^4=1\rangle$ be a cyclic group of order four. $K=(0,1)$ be a field of order 2. Then $1+g+g^2+g^3$ is not an idempotent. For

$$\begin{aligned}(1+g+g^2+g^3)^2 &= 1+g^2+g^4+g^6\\ &= 1+g^2+1+g^2\\ &= 2+2g^2\\ &= 0.\end{aligned}$$

In fact $(1+g+g^2+g^3)$ becomes a zero divisor.

THEOREM 5. *Let $G=\langle g \mid g^{2n}=1\rangle$ be a group of even order.* $K=(0,1)$ *be a field* *of two element. Then* G^* *contains* $1+g+g^2+\cdots+g^n+\cdots+g^{2n-1}$ *as a nontrivial zero divisor.*

PROOF. Clearly G^* is a semi-group. To show $1+g+g^2+\cdots+g^n+\cdots+g^{2n-1}$ is a zero divisor in G^*.

Consider

$$\begin{aligned}(1+g+g^2+\cdots+g^n+\cdots g^{2n-1})^2 &= (1+g^2+g^4+\cdots+1+\cdots+g^2+g^4+\cdots+g^{2n-1}) \\ &= 2(1+g^2+g^4+\cdots+g^{2n-2}) \\ &= 0\end{aligned}$$

(Since $g^{n+1}\cdot g^{n+1}=g^2$ and so on).

Clearly $1+g+g^2+\cdots+g^n+\cdots+g^{2n-1}$ is a zero divisor in G^*.

Now we pose the following problem.

PROBLEM. Let G be any cyclic group. K any field having p elements such that p does not divide the order of G.

What can be said about the semigroup G^*?

(1) Does G^* contain idempotent elements?

(2) Does G^* contain zero divisors?

(3) What is the order of G^*?

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REFERENCE

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