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ON M LOOPS

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1. Introduction

In a loop $\langle G, \cdot\rangle$, the following identities are known to be equivalent:

$$M^1 : xy\cdot zx=(x\cdot yz)x \ ; \ M^2 : (yx\cdot z)x=y(x\cdot zx) \ ; \ M^3 : x(y\cdot xz)=(xy\cdot x)z.$$

Loops satisfying these identities are called Moufang loops.

We observe that x appears twice on both sides of each of the identities. We are interested in another set of new identities formed by replacing one x on each side by x^θ where θ is a map of G. There are altogether twelve of them as listed below:

$$M^1_{\lambda\lambda} : x^\theta y\cdot zx=(x^\theta\cdot yz)x \quad M^2_{\lambda\lambda} : (yx^\theta\cdot z)x=y(x^\theta\cdot zx) \quad M^3_{\lambda\lambda} : x^\theta(y\cdot xz)=(x^\theta y\cdot x)z$$

$$M^1_{\rho\rho} : xy\cdot zx^\theta=(x\cdot yz)x^\theta \quad M^2_{\rho\rho} : (yx\cdot z)x^\theta=y(x\cdot zx^\theta) \quad M^3_{\rho\rho} : x(y\cdot x^\theta z)=(xy\cdot x^\theta)z$$

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Hala Orlik Pflugfelder has proven that a loop G satisfying $M^2_{\rho\rho}$ is a Moufang loop and that $x^{-1}x^\theta\in N$, the nucleus of G [2]. She calls loops with this identity as M loops.

Pl. Kannappan has shown that the relations $M^i_{\rho\rho}$, $i=1,2,3$, in a loop are equivalent [3].

This paper proves that the identities $M^i_{\lambda\lambda}$ and $M^i_{\rho\rho}$, $i=1,2,3$ in a loop are equivalent. The identities $M^i_{\lambda\rho}$ and $M^i_{\rho\lambda}$, $i=1,2,3$ in a loop are also equivalent if θ is a onto map. Loop satisfying the identities $M^i_{\lambda\rho}$ and $M^i_{\rho\lambda}$ are M loops. Moreover, they possess the additional property of $x^{-1}.\ x^\theta\in Z$, the centre of G.

2. Loops satisfying identities $M^i_{\lambda\lambda}$ or $M^i_{\rho\rho}$

LEMMA 1. *A loop $\langle G, \cdot\rangle$ satisfying the identity $M^i_{\lambda\lambda}$ or $M^i_{\rho\rho}$ possesses the I.P. (inverse property).*

PROOF. (a) $M^1_{\lambda\lambda}: x^\theta y\cdot zx=(x^\theta\cdot yz)x$

Define ${}^{-1}x$ by ${}^{-1}x\cdot x=1$. Let $z={}^{-1}x$. Then $x^\theta y=(x^\theta\cdot y^{-1}x)x$. It is clear from $M^1_{\lambda\lambda}$ that $x^\theta\cdot zx=x^\theta z\cdot x$. Thus $x^\theta y=x^\theta(y^{-1}x\cdot x)$ and $y=y^{-1}x\cdot x$.

Let $y=x$. We have $x\cdot{}^{-1}x=1$. Hence G has the R. I. P. (right inverse property).

Let $y=x^{-1}z^{-1}: x^\theta(x^{-1}z^{-1})\cdot zx=x^\theta$ by R. I. P.. So $(zx)^{-1}=x^{-1}z^{-1}$ by R. I. P. again.

Let $z^{-1}=x^\theta y: z^{-1}\cdot zx=(z^{-1}y^{-1}\cdot yz)x=(yz)^{-1}yz\cdot x=x$. Hence G has the L. I. P. (left inverse property).

(b) $M^2_{\lambda\lambda}: (yx^\theta\cdot z)x=y(x^\theta\cdot zx)$

Define ${}^{-1}x$ by ${}^{-1}x\cdot x=1$. Let $z={}^{-1}x$. Then $(yx^\theta\cdot{}^{-1}x)x=yx^\theta$. Put $yx^\theta=x$, we get $x\cdot{}^{-1}x=1$. Hence G has the R. I. P..

Let $y=z^{-1}(x^\theta)^{-1}$. $\therefore\ x=z^{-1}(x^\theta)^{-1}\cdot(x^\theta z\cdot x)$ since $x^\theta\cdot zx=x^\theta z\cdot x$ from $M^2_{\lambda\lambda}$. Thus G has the L. I. P..

(c) $M^3_{\lambda\lambda}: x^\theta(y\cdot xz)=(x^\theta y\cdot x)z$

Define x^{-1} by $x\cdot x^{-1}=1$. Let $z=x^{-1}$. Then $x^\theta y=(x^\theta y\cdot x)x^{-1}$. Put $x^\theta y=x^{-1}$. So $x^{-1}\cdot x=1$. Thus G has the R. I. P.. Let $y=x^{-1}$.

$\therefore\ x^\theta(x^{-1}\cdot xz)=x^\theta z\ \therefore\ x^{-1}\cdot xz=z$. So G has the R. I. P..

(d) $M^1_{\rho\rho}: xy\cdot zx^\theta=(x\cdot yz)x^\theta$

See [3].

(e) $M^2_{\rho\rho}: (yx\cdot z)x^\theta=y(x\cdot zx^\theta)$

See [3].

(f) $M^3_{\rho\rho}: x(y\cdot x^\theta z)=(xy\cdot x^\theta)z$

Define x^{-1} by $x\cdot x^{-1}=1$. Let $y=x^{-1}$.

$\therefore\ x(x^{-1}\cdot x^\theta z)=x^\theta z$. Let $x^\theta z=x$. $\therefore\ x^{-1}\cdot x=1$.

So G has the L. I. P.. From $M^3_{\rho\rho}$, $x\cdot yx^\theta=xy\cdot x^\theta$.

So $x(y\cdot x^\theta z)=(x\cdot yx^\theta)z$. Let $z=(x^\theta)^{-1}y^{-1}$.

$\therefore\ x=(x\cdot yx^\theta)(x^{\theta^{-1}}y^{-1})$ by the L. I. P.. So G has the R. I. P..

REMARK. In proving case (*f*), θ is required to be onto in [3, p. 105]. Apparently we don't need it here.

THEOREM 1. *The identities $M^i_{\lambda\lambda}$ and $M^i_{\rho\rho}$, $i=1,2,3$, for a loop $\langle G,\cdot\rangle$ are equivalent.*

PROOF. We use the autotopism lemma which says that if (U, V, W) is an autotopism of an I. P. loop, then (JUJ, V, W) and (W, JVJ, U) are also autotopisms where J is the inverse mapping. Also (U^{-1}, V^{-1}, W^{-1}) is also an autotopism [1, p. 112]. Note that $JL(x)J=R(x^{-1})$ and $JR(x)J=L(x^{-1})$.

$$M^1_{\lambda\lambda}\Leftrightarrow(L(x^\theta),\ R(x),\ L(x^\theta)R(x))\Leftrightarrow(L(x^\theta)R(x),\ L(x^{-1}),\ L(x^\theta))$$

$$\Updownarrow \qquad \Leftrightarrow M^3_{\lambda\lambda}\text{ (replacing } x^{-1}z \text{ by } z).$$

$$(R(x^\theta)^{-1},\ L(x^\theta)(R(x),\ R(x))\Leftrightarrow M^2_{\lambda\lambda}\text{ (replacing } yx^{\theta^{-1}} \text{ by } y).$$

$$\Updownarrow$$

$$(R(x^\theta),\ R(x^{-1})L(x^\theta)^{-1},\ R(x^{-1}))\Leftrightarrow(R(x^{-1}),\ L(x)R(x^\theta),\ R(x^\theta))$$

$$\Leftrightarrow M^2_{\rho\rho}\text{ (replacing } yx^{-1} \text{ by } y).$$

The proof is completed with [3, Theorem 1, 2, 3].

3. Loops satisfying identities $M^i_{\lambda\rho}$ or $M^i_{\rho\lambda}$

LEMMA 2. $M^1_{\lambda\rho}\Rightarrow M'_{\lambda\lambda}$; $M^1_{\rho\lambda}\Rightarrow M^1_{\rho\rho}$.

PROOF. Put $y=1$ in $M^1_{\lambda\rho}$. Then $x^\theta z\cdot x=xz\cdot x^\theta$.

$\therefore\ M^1_{\lambda\rho}\Rightarrow x^\theta z\cdot yx=(x\cdot zy)x^\theta=(x^\theta\cdot zy)x\Rightarrow M^1_{\lambda\lambda}$.

Put $z=1$ in $M^1_{\rho\lambda}$. Then $xy\cdot x^\theta=x^\theta y\cdot x$.

$\therefore\ M^1_{\rho\lambda}\Rightarrow xy\cdot zx^\theta=(x^\theta\cdot yz)x=(x\cdot yz)x^\theta\Rightarrow M^1_{\rho\rho}$.

LEMMA 3. $M^2_{\lambda\rho}\Rightarrow M^1_{\lambda\lambda}$ *if θ is onto.*

PROOF. $M^2_{\lambda\rho}$: $(yx^\theta\cdot z)x=y(x^\theta z\cdot x)$. Define $(x^\theta)^{-1}$ by $x^\theta(x^\theta)^{-1}=1$.

Let $z=(x^\theta)^{-1}$. $\therefore$ $(yx^\theta\cdot(x^\theta)^{-1})x=yx$. So $yx^\theta\cdot(x^\theta)^{-1}=y$.

Let $y=(x^\theta)^{-1}$. Then $(x^\theta)^{-1}\cdot x^\theta=1$. Thus it has the R. I. P..

Let $y=z^{-1}(x^\theta)^{-1}$. $\therefore$ $x=(z^{-1}(x^\theta)^{-1})(x^\theta z\cdot x)$ by using the R. I. P.. Thus it has the L. I. P..

Replacing yx^θ by y, we have $yz(R(x)=yR(x^{\theta^{-1}})z\ L(x^\theta)R(x)$.

$\therefore$ $(R(x^\theta)^{-1},\ L(x^\theta)R(x),\ R(x))$ is an autotopism.

$\therefore$ $(L(x^\theta),\ R(x),\ L(x^\theta)R(x))$ is an autotopism.

$\therefore$ $x^\theta y\cdot zx=(x^\theta\cdot yz)x$.

$\therefore$ $M^2_{\lambda\rho}\Rightarrow M^1_{\lambda\lambda}$.

LEMMA 4. $M^2_{\rho\lambda}\Rightarrow M^1_{\rho\rho}$ *if θ is onto.*

PROOF. $M^2_{\rho\lambda}$: $(yx\cdot z)x^\theta=y(x^\theta\cdot zx)$.

$y=1 \Rightarrow xz\cdot x^\theta = x^\theta \cdot zx$. $\therefore$ $(yx\cdot z)x^\theta = y(x^\theta \cdot zx) = y(xz\cdot x^\theta)$.
Define x^{-1} by $x\cdot x^{-1}=1$. Let $z=x^{-1}$. $\therefore$ $(yx\cdot x^{-1})x^\theta = yx^\theta$.
$\therefore$ $yx\cdot x^{-1}=y$. Let $y=x^{-1}$. Then $x^{-1}\cdot x=1$. Thus it has the R.I.P..
Let $y=z^{-1}x^{-1}$. $\therefore$ $x^\theta = z^{-1}x^{-1}\cdot(xz\cdot x^\theta)$ by using the R.I.P..
$\therefore$ It has the L.I.P..
Replacing yx by y, we have $yzR(x^\theta) = yR(x^{-1})zL(x)R(x^\theta)$.
$\therefore$ $(R(x^{-1}),\ L(x)R(x^\theta),\ R(x^\theta))$ is an autotopism.
$\therefore$ $(L(x),\ R(x^\theta),\ L(x)R(x^\theta))$ is an autotopism.
$\therefore$ $xy\cdot zx^\theta = (x\cdot yz)x^\theta$.
$\therefore$ $M^2_{\rho\lambda} \Rightarrow M^1_{\rho\rho}$.

LEMMA 5. $M^3_{\lambda\rho} \Rightarrow M^1_{\lambda\rho}$ *if* θ *is onto.*

PROOF. $M^3_{\lambda\rho}$: $x^\theta(y\cdot xz) = (xy\cdot x^\theta)z$
Define x^{-1} by $x\cdot x^{-1}=1$. Let $y=x^{-1}$. $\therefore$ $x^\theta(x^{-1}\cdot xz) = x^\theta z$.
$\therefore$ $x^{-1}\cdot xz=z$. Let $z=x^{-1}$. Thus $x^{-1}\cdot x=1$.
$\therefore$ It has the L.I.P.. Let $z=x^{-1}y^{-1}$.
Then $x^\theta = (x^\theta\cdot yx)(x^{-1}y^{-1})$ using the L.I.P. and $xy\cdot x^\theta = x^\theta\cdot yx$.
$\therefore$ It has the R.I.P..
Replacing xz by z, we have $yzL(x^\theta) = yL(x)R(x^\theta)zL(x^{-1})$.
$\therefore$ $(L(x)R(x^\theta),\ L(x^{-1}),\ L(x^\theta))$ is an autotopism.
$\therefore$ $(L(x^\theta),\ R(x),\ L(x)R(x^\theta))$ is an autotopism.
$\therefore$ $x^\theta y\cdot zx = (x\cdot yz)x^\theta$.
$\therefore$ $M^3_{\lambda\rho} \Rightarrow M^1_{\lambda\rho}$.

LEMMA 6. $M^3_{\rho\lambda} \Rightarrow M^1_{\rho\lambda}$ *if* θ *is onto.*

PROOF. $M^3_{\rho\lambda}$: $x(y\cdot x^\theta z) = (x^\theta y\cdot x)z = (x\cdot yx^\theta)z$.
Define ${}^{-1}(x^\theta)$ by ${}^{-1}(x^\theta)\cdot x^\theta=1$. Let $y={}^{-1}(x^\theta)$.
Then $x({}^{-1}(x^\theta)\cdot x^\theta z) = xz$. $\therefore$ ${}^{-1}(x^\theta)\cdot x^\theta z = z$.
Let $z={}^{-1}(x^\theta)$. $\therefore$ $x^\theta\cdot{}^{-1}(x^\theta)=1$. $\therefore$ It has the L.I.P..
Let $z=(x^\theta)^{-1}y^{-1}$. Then $x=(x\cdot yx^\theta)(x^\theta y^{-1})$ using the L.I.P..
$\therefore$ It has the R.I.P..
Replacing $x^\theta z$ by z, we have $yzR(x) = yL(x^\theta)R(x)\cdot zL(x^\theta)^{-1}$.
$\therefore$ $(L(x^\theta)R(x),\ L(x^\theta)^{-1},\ R(x))$ is an autotopism.

$\therefore$ $(R(x),\ R(x^{\theta}),\ L(x^{\theta})R(x))$ is an autotopism.

$\therefore$ $yx\cdot zx^{\theta}=x^{\theta}(yz)\cdot x$.

$\therefore$ $M^{3}_{\rho\lambda}\Rightarrow M^{1}_{\rho\lambda}$.

THEOREM 2. *Let G be a loop with the identity* $M^{i}_{\lambda\rho}$ *or* $M^{i}_{\rho\lambda}$ $(i=1,2,3)$. *Then* $x^{-1}\cdot x^{\theta}\in Z\ \forall x\in G$.

PROOF. From lemma 2 to lemma 6, we see that $M^{i}_{\lambda\rho}\Rightarrow M^{1}_{\lambda\lambda}$ and $M^{i}_{\rho\lambda}\Rightarrow M^{1}_{\rho\rho}$. Therefore, $x^{-1}\cdot x^{\theta}\in N\ \forall x\in G$. By Moufang's theorem, [1, p. 117], $\langle x^{\theta}, x, y\rangle$ is a group $\forall y\in G$. By putting $z=1$ in $M^{i}_{\lambda\rho}$ or $M^{i}_{\rho\lambda}$, we get $x^{\theta}yx=xyx^{\theta}$. Replacing y by yx^{-1}, we have $x^{-1}x^{\theta}\cdot y=y\cdot x^{-1}x^{\theta}$. $\therefore$ $x^{-1}x^{\theta}\in Z$.

THEOREM 3. *The identities* $M^{i}_{\lambda\rho}$ *and* $M^{i}_{\rho\lambda}$ *in a loop G are equivalent,* $i=1,2,3$.

PROOF. By Theorem 2, $x^{-1}x^{\theta}\in Z\ \forall x\in G$. Then $x^{\theta}=xz_{x}$, $z_{x}\in Z$. Using this and the equivalence of Moufang identities M^{i}, $i=1,2,3$, we can complete the proof easily.

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